A new polynomial algorithm for nested resource allocation, speed optimization and other related problems

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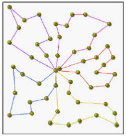
Timing problems in vehicle routing

- General effort dedicated to better address rich vehicle routing problems involving many side constraints and attributes
- Observation : many rich VRPs are hard because of their time features: (single, soft, or multiple) time windows, time-dependent, flexible or stochastic travel times, various time-dependent costs, break scheduling...
- Timing subproblems: similar formulations in various domains: VRP, scheduling, PERT, resource allocation, isotone regression, telecommunications...
- Cross-domain analysis of timing problems and algorithms:
  - T. Vidal, T. G. Crainic, M. Gendreau, and C. Prins. A Unifying View on Timing Problems and Algorithms. Submitted & revised to Networks. Tech. Rep. CIRRELT 2011-43.

- Four different applications

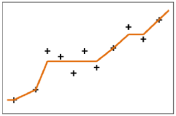
VRPTW



E/T scheduling

ship speed opt.

isotonic regression



- **VRP with soft time windows**. Optimizing arrival times for a given sequence of visits σ:

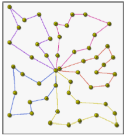
$$\min_{\mathbf{t}\geq\mathbf{0}} \ \alpha \sum_{i=1}^{|\sigma|} \max\{e_{\sigma(i)} - t_{\sigma(i)}, 0\} + \beta \sum_{i=1}^{|\sigma|} \max\{t_{\sigma(i)} - l_{\sigma(i)}, 0\} \tag{1.1}$$

$$\text{s.t.} \quad t_{\sigma(i)} + \delta_{\sigma(i)\sigma(i+1)} \leq t_{\sigma(i+1)} \qquad 1 \leq i < |\sigma| \tag{1.2}$$

Some examples

- Four different applications

VRPTW



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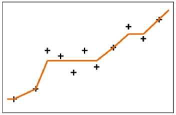


- **E/T scheduling**. Optimizing processing dates for a given sequence of visits σ:

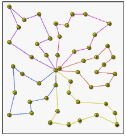
$$\min_{\mathbf{t}\geq\mathbf{0}} \sum_{i=1}^{|\sigma|} \alpha_i \max\{d_{\sigma(i)} - t_{\sigma(i)}, 0\} + \sum_{i=1}^{|\sigma|} \beta_i \max\{t_{\sigma(i)} - d_{\sigma(i)}, 0\} \quad (1.3)$$

$$\text{s.t.} \quad t_{\sigma(i)} + p_{\sigma(i)} \leq t_{\sigma(i+1)} \qquad 1 \leq i < |\sigma| \quad (1.4)$$

Some examples

- Four different applications

VRPTW



E/T scheduling

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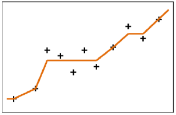


- **Ship speed optimization**. Optimizing leg speeds to visit a sequence of locations σ:

$$\min_{\mathbf{t} \geq \mathbf{0}} \sum_{i=1}^{|\sigma|-1} d_{\sigma(i)\sigma(i+1)} \hat{c} \left(\frac{d_{\sigma(i)\sigma(i+1)}}{t_{\sigma(i+1)} - t_{\sigma(i)}} \right) \tag{1.5}$$

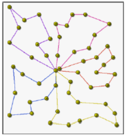
$$\text{s.t.} \quad t_{\sigma(i)} + p_{\sigma(i)} + \frac{d_{\sigma(i)\sigma(i+1)}}{v_{max}} \leq t_{\sigma(i+1)} \qquad 1 \leq i < |\sigma| \tag{1.6}$$

$$r_{\sigma(i)} \leq t_{\sigma(i)} \leq d_{\sigma(i)} \qquad 1 \leq i \leq |\sigma| \tag{1.7}$$

Some examples

- Four different applications

VRPTW



E/T scheduling

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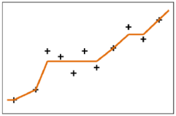


- **Isotonic Regression**. Given a vector $\mathbf{N} = (N_1, \ldots, N_n)$ of n real numbers, finding a vector of non-decreasing values $\mathbf{t} = (t_1, \ldots, t_n)$ *as close as possible* to $\mathbf{N}$ according to a distance metric:

$$\min_{\mathbf{t}=(t_1,\ldots,t_n)} \quad \|\mathbf{t} - \mathbf{N}\| \tag{1.8}$$

$$\text{s.t.} \quad t_i \leq t_{i+1} \qquad 1 \leq i < n \tag{1.9}$$

General Timing Problem

- Timing problems:

$$\min_{\mathbf{t} \geq \mathbf{0}} \quad \sum_{F^x \in \mathcal{F}^{\text{OBJ}}} \alpha_x \sum_{1 \leq y \leq m_x} f_y^x(\mathbf{t}) \tag{1.10}$$

$$s.t. \quad t_i + p_i \leq t_{i+1} \qquad 1 \leq i < n \tag{1.11}$$

$$f_y^x(\mathbf{t}) \leq 0 \qquad F^x \in \mathcal{F}^{\text{CONS}}, \ 1 \leq y \leq m_x \tag{1.12}$$

- Continuous variables t_i following a **total order**.
- **Additional features** characterized by functions $f_y^x(\mathbf{t})$ for $y \in \{1, \ldots, m_x\}$, either in the objective or as constraints.
- Many names in the literature: scheduling, timing, projection onto order simplexes, optimal service time problem...

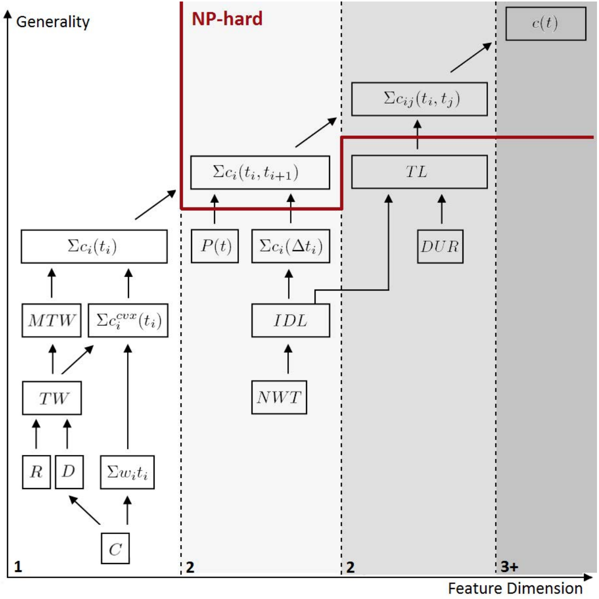
Features

- Rich vehicle routing problems can involve various timing features

Symbol	Parameters	Char. functions	ξ	Most frequent roles
W	Weights w_i	$f_i(\mathbf{t}) = w_i t_i$	1	Weighted execution dates
D	Deadlines d_i	$f_i(\mathbf{t}) = (t_i - d_i)^+$	1	Deadline constraints, tardiness
R	Release dates r_i	$f_i(\mathbf{t}) = (r_i - t_i)^+$	1	Release-date constraints, earliness.
TW	Time windows $TW_i = [r_i, d_i]$	$f_i(\mathbf{t}) = (t_i - d_i)^+ + (r_i - t_i)^+$	1	Time-window constraints, soft time windows.
MTW	Multiple TW $MTW_i = \cup[r_{ik}, d_{ik}]$	$f_i(\mathbf{t}) = \min_k [(t_i - d_{ik})^+ + (r_{ik} - t_i)^+]$	1	Multiple time-window constraints
$\Sigma c_i^{\text{cvx}}(t_i)$	Convex $c_i^{\text{cvx}}(t_i)$	$f_i(\mathbf{t}) = c_i^{\text{cvx}}(t_i)$	1	Separable convex objectives
$\Sigma c_i(t_i)$	General $c_i(t)$	$f_i(\mathbf{t}) = c_i(t_i)$	1	Separable objectives, time-dependent activity costs
DUR	Total dur. δ_{max}	$f(\mathbf{t}) = (t_n - \delta_{max} - t_1)^+$	2	Duration or overall idle time
NWT	No wait	$f_i(\mathbf{t}) = (t_{i+1} - p_i - t_i)^+$	2	No wait constraints, min idle time
IDL	Idle time ι_i	$f_i(\mathbf{t}) = (t_{i+1} - p_i - \iota_i - t_i)^+$	2	Limited idle time by activity, min idle time excess
$P(t)$	Time-dependent proc. times $p_i(t_i)$	$f_i(\mathbf{t}) = (t_i + p_i(t_i) - t_{i+1})^+$	2	Processing-time constraints, min activities overlap
TL	Time-lags δ_{ij}	$f_i(\mathbf{t}) = (t_j - \delta_{ij} - t_i)^+$	2	Min excess with respect to time-lags
$\Sigma c_i(\Delta t_i)$	General $c_i(t)$	$f_i(\mathbf{t}) = c_i(t_{i+1} - t_i)$	2	Separable functions of durations between successive activities, flex. processing times
$\Sigma c_{ij}(t_i, t_j)$	General $c_{ij}(t, t')$	$f_{ij}(\mathbf{t}) = c_i(t_i, t_j)$	2	Separable objectives or constraints by any pairs of variables

Hierarchy of features

- These features can be classified within a hierarchy (using many-one linear reduction relationships between the associated timing problems)
- Features in the NP-hard area lead to NP-hard timing problems



Re-optimization

- Some particular features have been extensively studied in various fields.
  - For example for the problem $\{\Sigma c_i^{\text{CVX}}(t_i) | \varnothing\}$ 30 algorithms from various domains (routing, scheduling, PERT, isotonic regression) were inventoried, based on only three main concepts.
- Key lines of research related to the resolution of series of similar timing problems within neighborhood searches, considering different sequences σ.

$$\min_{\mathbf{t} \geq \mathbf{0}} \quad \sum_{F^x \in \mathcal{F}^{\text{OBJ}}} \alpha_x \sum_{1 \leq y \leq m_x} f_y^x(\mathbf{t}) \tag{1.13}$$

$$s.t. \quad t_{\sigma^k(i)} + p_{\sigma^k(i),\sigma^k(i+1)} \leq t_{\sigma^k(i+1)} \qquad 1 \leq i < |\sigma| \tag{1.14}$$

$$f_y^x(\mathbf{t}) \leq 0 \qquad F^x \in \mathcal{F}^{\text{CONS}}, \ 1 \leq y \leq m_x \tag{1.15}$$

Contents

One particular problem

- Consider one particular timing problem with flexible travel times and deadlines:

$$\min_{\mathbf{t} \geq \mathbf{0}} \sum_{i=1}^{|\sigma|-1} c_i(t_{\sigma(i+1)} - t_{\sigma(i)}) \tag{2.1}$$

$$\text{s.t.} \quad t_{\sigma(i)} + p_{\sigma(i)} + \frac{d_{\sigma(i)\sigma(i+1)}}{v_{max}} \leq t_{\sigma(i+1)} \qquad 1 \leq i < |\sigma| \tag{2.2}$$

$$t_{\sigma(i)} \leq d_{\sigma(i)} \qquad 1 \leq i \leq |\sigma| \tag{2.3}$$

$$t_{\sigma(|\sigma|)} = B \tag{2.4}$$

- It is a vehicle speed optimization problem with convex – **and possibly heterogeneous** – cost/speed functions per leg.
- Direct applications related to:
  - Ship speed optimization (Norstad et al., 2011; Hvattum et al., 2013)
  - Vehicle routing with flexible travel time or pollution routing (Hashimoto et al., 2006; Bektas and Laporte, 2011)

One particular problem

- Consider one particular timing problem with flexible travel times and deadlines:

$$\min_{\mathbf{t}\geq\mathbf{0}} \sum_{i=1}^{|\sigma|-1} c_i(t_{\sigma(i+1)} - t_{\sigma(i)}) \tag{2.5}$$

$$\text{s.t.} \quad t_{\sigma(i)} + p_{\sigma(i)} + \frac{d_{\sigma(i)\sigma(i+1)}}{v_{max}} \leq t_{\sigma(i+1)} \qquad 1 \leq i < |\sigma| \tag{2.6}$$

$$t_{\sigma(i)} \leq d_{\sigma(i)} \qquad 1 \leq i \leq |\sigma| \tag{2.7}$$

$$t_{\sigma(|\sigma|)} = B \tag{2.8}$$

- A quick reformulation
  - Waiting times can be modeled by additional activities with null cost
  - Change of variables $x_i = t_{\sigma(i+1)} - t_{\sigma(i)} - p_{\sigma(i)} - \frac{d_{\sigma(i)\sigma(i+1)}}{v_{max}}$
  - leads to...

A resource allocation problem

- A resource allocation problem with nested constraints (NESTED)

$$\min \quad f(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i) \tag{2.9}$$

$$\text{s.t.} \quad 0 \leq x_i \leq d_i \qquad i \in \{1, \dots, n\} \tag{2.10}$$

$$\sum_{k=1}^{s[i]} x_k \leq a_i \qquad i \in \{1, \dots, m-1\} \tag{2.11}$$

$$\sum_{i=1}^{n} x_i = B \tag{2.12}$$

  - **Integer or continuous variables** are considered here
  - Travel time x_i on each leg, subject to a maximum bound d_i.
  - Deadlines a_i on arrival time at some ports.
  - Table $s[]$ listing the indices of variables on which deadlines are applied. There may be less deadline constraints m than variables n.
  - Final arrival date B.

A resource allocation problem

- **Without the nested constraints** (2.16) ⇒ Standard resource allocation problem (Ibaraki and Katoh, 1988; Patriksson, 2008)

$$\min_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{d}} \quad f(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i) \tag{2.13}$$

$$\text{s.t.} \sum_{i=1}^{n} x_i = B \tag{2.14}$$

  - Interesting applications to search-effort allocation, portfolio selection, energy optimization, sample allocation in stratified sampling, capital budgeting, mass advertising, and matrix balancing, among others.

A resource allocation problem

- Various applications

$$\min_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{d}} \quad f(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i) \tag{2.15}$$

$$\text{s.t.} \sum_{k=1}^{s[i]} x_k \leq a_i \qquad i \in \{1, \ldots, m-1\} \tag{2.16}$$

$$\sum_{i=1}^{n} x_i = B \tag{2.17}$$

- **With the nested constraints**, additional applications to
  - Project crashing (Talbot, 1982)
  - Production and resource planning (Bellman et al., 1954; Bellman and Dreyfus, 1962; Veinott, 1964)
  - Lot sizing (Tamir, 1980)
  - Assortment with downward substitution (Hanssmann, 1957; Sadowski, 1959; Pentico, 2008)
  - Telecommunications (Padakandla and Sundaresan, 2009a)

ϵ-approximate solutions

- Computational complexity of algorithms for general non-linear optimization problems $\Rightarrow$ an infinite output size may be needed due to real optimal solutions.

- To circumvent this issue
  - Existence of an oracle which returns the value of $f_i(x)$ in $O(1)$
  - Approximate notion of optimality (Hochbaum and Shanthikumar, 1990):

    a continuous solution $\mathbf{x}^{(\epsilon)}$ is ϵ-accurate iff there exists an optimal solution $\mathbf{x}^*$ such that $||(\mathbf{x}^{(\epsilon)} - \mathbf{x}^*)||_\infty \leq \epsilon$.

  - Accuracy is defined in the solution space, in contrast with some other approximation approaches which considered objective space (Nemirovsky and Yudin, 1983).

Existing algorithms – VRP or ship routing literature

- Recursive smoothing algorithm (Norstad et al., 2011; Hvattum et al., 2013)
  - **Applicable only when the cost/speed functions are arc-independent**
  - This case is strongly polynomial (which even never needs to evaluate the objective function)
  - Complexity : $O(n^2)$

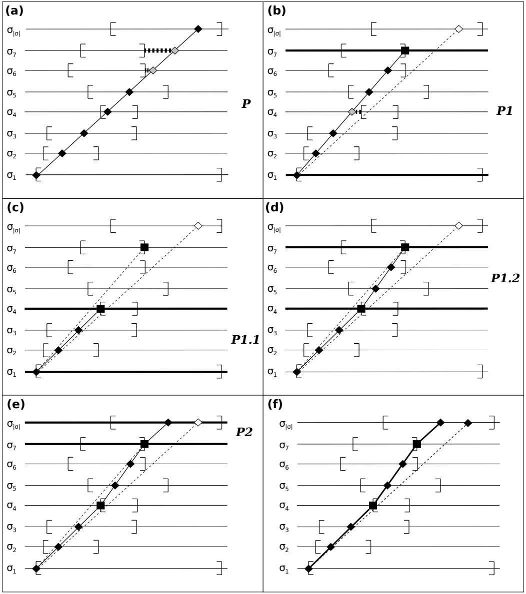


Image from R. Kramer, A. Subramanian, T. Vidal, and L. A. F. Cabral. *A matheuristic approach for the Pollution-Routing Problem.* 2014. arXiv: 1404.4895v1

Existing algorithms – VRP or ship routing literature

- And this approach is closely related to the concept of *string method* (Dantzig 1971 and other earlier contributions)

(23)

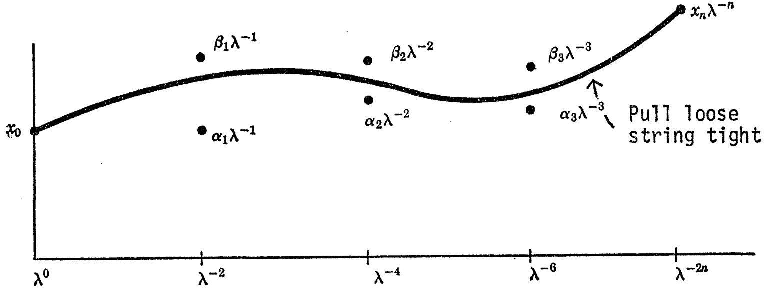


Image from G. B. Dantzig. A control problem of Bellman. Management Science. 17(9), pp. 542–546, 1971.

Existing algorithms – VRP or ship routing literature

- Dynamic programming approach for the case of piecewise linear and convex functions (Hashimoto et al., 2006)
- Compute recursively the functions $F_i(b)$ which evaluate the minimum cost to execute the i first activities $(x_1, \ldots, x_i)$ with a resource consumption of b.
- *Bi-directional dynamic programming can be used. An efficient way to solve serial problems with different (but similar) sequences, using pre-processing and incremental evaluation of moves.*

Existing algorithms – Others

- **Dual-inspired methods**. Rely on the fact that the continuous resource allocation problem without nested constraints (2.16) can be solved by finding the zero of a single Lagrangian equation:

$$
\begin{aligned}
L'_{\mathrm{RAP}}(\lambda) = \sum_{i=v}^{w} \bar{x}_i(\lambda) - B = 0 \\
\text{with} \quad \bar{x}_i(\lambda) = {f'}_i^{-1}\left(\max({f'}_i(0), \min(\lambda, {f'}_i(d_i)))\right)
\end{aligned}
\tag{2.18}
$$

- Iteratively solving Lagrangian equations and adjusting violated nested constraints by variable setting.
  - Padakandla and Sundaresan (2009a): complexity of $O(n^2 \Phi_{\mathrm{RAP}}(n, B))$
  - Wang (2014): complexity of $O(n^2 \log n + n\Phi_{\mathrm{RAP}}(n, B))$
  - where $\Phi_{\mathrm{RAP}}(n, B)$ is the complexity of solving one RAP with n tasks, e.g., by bisection search.

Existing algorithms – Others

- A greedy method with scaling for NESTED with integer variables (Hochbaum, 1994)
  - **Greedy** algorithms iteratively consider all feasible increments of one resource, and select the least-cost one.
  - **Convergence guarantee** (Federgruen and Groenevelt, 1986) to the optimum of the integer RAP in the presence of polymatroidal constraints.

- **Scaling**.
  - An initial problem is solved with large increments
  - The increment size is iteratively divided by two to achieve higher accuracy.
  - At each iteration, and for each variable, only one increment from the previous iteration may require to be corrected.
  - Complexity of $O(n \log n \log \frac{B}{n})$ for NESTED with integer variables

Proximity theorem

- **Proximity Theorem** (Hochbaum, 1994):

Theorem

For any optimal continuous solution $\mathbf{x}$ *of NESTED, there exists an optimal solution* $\mathbf{z}$ *of the same problem with integer variables, such that* $\mathbf{z} - \mathbf{e} < \mathbf{x} < \mathbf{z} + n\mathbf{e}$*, and thus* $||\mathbf{z} - \mathbf{x}||_\infty \leq n$*. Reversely, for any integer optimal solution* $\mathbf{z}$*, there exists an optimal continuous solution such that* $||\mathbf{z} - \mathbf{x}||_\infty \leq n$*.*

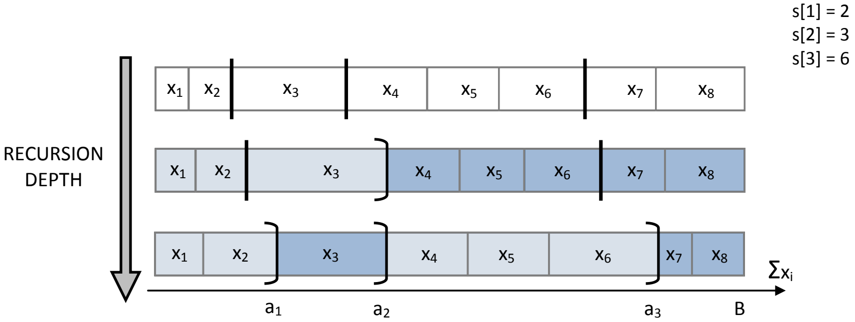
Corollary

To obtain an ϵ*-approximate solution of the NESTED problem with continuous variables, it is possible to solve a scaled NESTED problem with integer variables, in which all problem parameters have been multiplied by* $\lceil \frac{n}{\epsilon} \rceil$*.*

Contents

Proposed Algorithm

- Simple divide and conquer framework: to solve a $\textsc{Nested}(v, w)$ subproblem, first solve $\textsc{Nested}(v, t)$ and $\textsc{Nested}(t + 1, w)$, and use this information to solve more efficiently the original problem.
- But how to use the information from subproblems...



Proposed Algorithm

- First an initialization step and feasibility check, then the main loop of the algorithm is the following:

Algorithm 1 Nested(v, w)

if $v = w$ **then**
  $(x_{s[v-1]+1}, \ldots, x_{s[v]}) \leftarrow$ Rap(v, v)
else
  Solve two subproblems:
  $t \leftarrow \lfloor \frac{v+w}{2} \rfloor$
  $(x_{s[v-1]+1}, \ldots, x_{s[t]}) \leftarrow$ Nested(v, t)
  $(x_{s[t]+1}, \ldots, x_{s[w]}) \leftarrow$ Nested$(t+1, w)$

  Do something to solve the upper level:
  for $i = s[v-1]+1$ to $s[t]$ **do**
    $(\bar{c}_i, \bar{d}_i) \leftarrow (0, x_i)$
  for $i = s[t]+1$ to $s[w]$ **do**
    $(\bar{c}_i, \bar{d}_i) \leftarrow (x_i, d_i)$
  $(x_{s[v-1]+1}, \ldots, x_{s[w]}) \leftarrow$ Rap(v, w)

Proposed Algorithm

- Claim: the algorithm $\textsc{Nested}(v, w)$ is a valid divide-and-conquer approach which returns the optimal solution of the following model:

$$
\textsc{Nested}(v, w)
\begin{cases}
\min & \displaystyle\sum_{i=s[v-1]+1}^{s[w]} f_i(x_i) \\
\text{s.t.} & \displaystyle\sum_{k=s[v-1]+1}^{s[i]} x_k \leq \ \bar{a}_i - \bar{a}_{v-1} \quad\quad i \in \{v, \ldots, w-1\} \\
& \displaystyle\sum_{i=s[v-1]+1}^{s[w]} x_i = \ \bar{a}_w - \bar{a}_{v-1} \\
& 0 \leq x_i \leq \ d_i \quad\quad\quad\quad\quad\quad\quad i \in \{s[v-1]+1, \ldots, s[w]\}
\end{cases}
$$

Proposed Algorithm

- $\textsc{Rap}(v, w)$ is a simple resource allocation problem with updated bounds.

$$\textsc{Rap}(v,w)\begin{cases}\min & \displaystyle\sum_{i=s[v-1]+1}^{s[w]} f_i(x_i) \\ \text{s.t.} & \displaystyle\sum_{i=s[v-1]+1}^{s[w]} x_i = \bar{a}_w - \bar{a}_{v-1} \\ & \hat{c}_i \leq x_i \leq \hat{d}_i \qquad\qquad i \in \{s[v-1]+1, \dots, s[w]\}\end{cases}$$

- Any classic method can be used to solve this problem.
  - Integer variables : $O(n \log \frac{B}{n})$ by Frederickson and Johnson (1982)
  - Continuous variables : can use bisection search on the Lagrangian dual

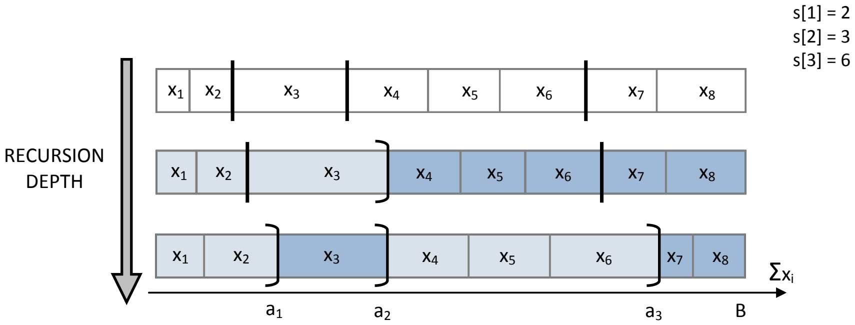
Convergence

Theorem

Consider (v, t, w) *s.t.* $1 \leq v \leq t \leq w \leq m$ *and* $v < w$. *Let* $(x^{\downarrow *}_{s[v-1]+1}, \ldots, x^{\downarrow *}_{s[t]})$ *and* $(x^{\uparrow *}_{s[t]+1}, \ldots, x^{\uparrow *}_{s[w]})$ *be optimal integer solutions of* $\textsc{Nested}(v, t)$ *and* $\textsc{Nested}(t+1, w)$, *then* $\textsc{Nested}(v, w)$ *admits an optimal integer solution* $(x^{**}_{s[v-1]+1}, \ldots, x^{**}_{s[w]})$ *such that*

$$x^{**}_i \leq x^{\downarrow *}_i \qquad i \in \{s[v-1]+1, \ldots, s[t]\} \tag{3.1}$$

$$x^{**}_i \geq x^{\uparrow *}_i \qquad i \in \{s[t]+1, \ldots, s[w]\} \tag{3.2}$$



Convergence

Theorem

Consider (v, t, w) s.t. $1 \leq v \leq t \leq w \leq m$ and $v < w$. Let $(x^{\downarrow *}_{s[v-1]+1}, \dots, x^{\downarrow *}_{s[t]})$ and $(x^{\uparrow *}_{s[t]+1}, \dots, x^{\uparrow *}_{s[w]})$ be optimal integer solutions of $\textsc{Nested}(v, t)$ *and* $\textsc{Nested}(t+1, w)$*, then* $\textsc{Nested}(v, w)$ *admits an optimal integer solution $(x^{**}_{s[v-1]+1}, \dots, x^{**}_{s[w]})$ such that*

$$x^{**}_i \leq x^{\downarrow *}_i \qquad i \in \{s[v-1]+1, \dots, s[t]\} \tag{3.3}$$

$$x^{**}_i \geq x^{\uparrow *}_i \qquad i \in \{s[t]+1, \dots, s[w]\} \tag{3.4}$$

- The valid inequalities (3.3-3.4) can be added to the formulation of $\textsc{Nested}(v, w)$.
- Alone, they guarantee that nested constraints are satisfied
  $\Rightarrow$ nested constraints can thus be eliminated.
- This leads to a $\textsc{Rap}(v, w)$ with updated bounds which can be efficiently solved.

Convergence

- Proof of this theorem, in the integer case, using the properties of the greedy algorithm
- For continuous variables, use the proximity theorem of Hochbaum (1994) with a suitable scaling coefficient.
- Alternatively, the KKT conditions can be used for a different proof by contradiction, but need of strong convexity and differentiability (not needed in the first proof).

Complexity

Theorem

The proposed decomposition algorithm for NESTED with integer variables works with a complexity of $O(n \log m \log \frac{B}{n})$.

- In the continuous case, an ϵ-approximate solution is obtained in $O(n \log m \log \frac{B}{\epsilon})$ operations
- For quadratic NESTED, an overall complexity of $O(n \log m)$ is achieved, using Brucker (1984) or Maculan et al. (2003) for the quadratic RAP sub-problems

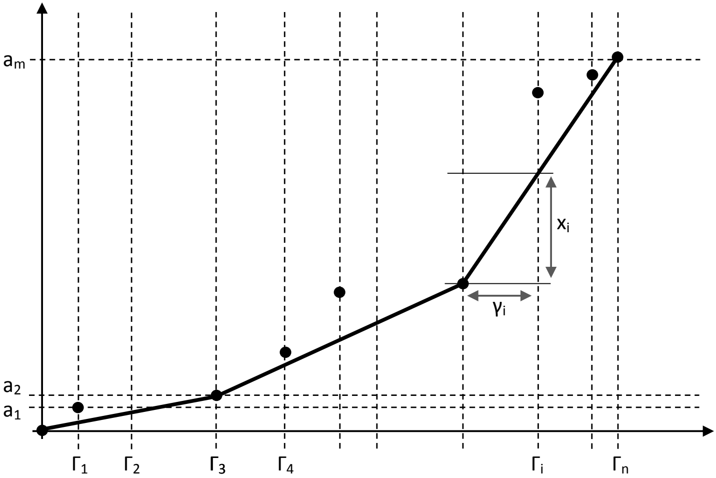
Contents

A remark on the expected number of active constraints

- Assume random-generated problem instances such that:
  - $d_i = +\infty$;
  - functions f_i strictly convex and differentiable, $f_i(x) = \gamma_i h(x/\gamma_i)$
- Define $\Gamma_i = \sum_{k=1}^{i} \gamma_k$ for $i \in \{0, \ldots, n\}$.
- We can show that solving the KKT conditions of NESTED under these assumptions is equivalent to computing the convex hull of the set of points $\mathcal{P}$ such that

$$\mathcal{P} = \{(\Gamma_{s[j]}, a_j) \mid j \in \{0, \ldots, m\}\}. \tag{4.1}$$

A remark on the expected number of active constraints



A remark on the expected number of active constraints

- Assume is addition that
  - $\alpha_i = a_{i+1} - a_i$ are i.i.d. random variables;
  - γ_i are i.i.d. random variables independent from the α_i's
  - and the vectors (γ_i, α_i) are non-colinear.

- Then the expected number of points on the convex hull grows as $O(\log m)$ (Baxter, 1961). Equivalently, there are **O(log m) expected active nested constraints in the solution**.

- This has a large practical impact when the complexity of the method depends on the number of active constraints

Contents

Metho 1

- To assess the practical performance of the proposed algorithm, we implemented it as well as the three other methods.
  - PS09 : dual algorithm of Padakandla and Sundaresan (2009b);
  - W14 : dual algorithm of Wang (2014);
  - H94 : scaled greedy algorithm of Hochbaum (1994);
  - MOSEK : interior point method of MOSEK (Andersen et al., 2003, for conic quadratic opt.);
  - THIS : proposed decomposition method.

- In these tests, we rely on a simple bisection search on the Lagrangian equation to solve the RAP subproblem.

Metho 1

- Each algorithm is tested on randomly-generated instances of NESTED problems (100 or 10 per type and size) with three families of objective functions.

$$[\text{F}] \qquad f_i(x) = \frac{x^4}{4} + p_i x \qquad x \in [0,1] \qquad (5.1)$$

$$[\text{Crashing}] \qquad f_i(x) = k_i + \frac{p_i}{x} \qquad x \in [c_i, d_i] \qquad (5.2)$$

$$[\text{FuelOpt}] \qquad f_i(x) = p_i \times c_i \times \left(\frac{c_i}{x}\right)^3 \qquad x \in [c_i, d_i] \qquad (5.3)$$

  - Size of instances ranges from $n = 10$ to $1,000,000$.
  - Accuracy of $\epsilon = 10^{-8}$
  - Coded in C++
  - Tests conducted on a Xeon 3.07 GHz CPU

Results $m = n$

Instance	n	nb Active	Time (s) PS09	W14	H94	MOSEK	THIS
[F]	10	1.15	8.86×10^{-5}	8.06×10^{-5}	6.18×10^{-5}	8.73×10^{-3}	1.85×10^{-5}
	10^2	1.04	7.96×10^{-3}	7.03×10^{-3}	6.74×10^{-4}	2.03×10^{-2}	1.69×10^{-4}
	10^4	1.15	1.06×10^{2}	8.72×10^{1}	1.46×10^{-1}	–	2.23×10^{-2}
	10^6	1.10	–	–	4.42×10^{1}	–	4.36
[F-Uniform]	10	2.92	1.03×10^{-4}	4.57×10^{-5}	5.86×10^{-5}	8.76×10^{-3}	2.62×10^{-5}
	10^2	5.06	1.37×10^{-2}	1.61×10^{-3}	7.42×10^{-4}	2.14×10^{-2}	4.97×10^{-4}
	10^4	9.99	–	6.08	1.67×10^{-1}	–	1.31×10^{-1}
	10^6	14.50	–	–	7.06×10^{1}	–	4.62×10^{1}
[F-Active]	10	3.67	1.19×10^{-4}	3.94×10^{-5}	5.76×10^{-5}	8.71×10^{-3}	2.88×10^{-5}
	10^2	10.00	2.28×10^{-2}	9.65×10^{-4}	7.50×10^{-4}	2.18×10^{-2}	4.69×10^{-4}
	10^4	50.75	–	2.31	1.62×10^{-1}	–	9.95×10^{-2}
	10^6	280.30	–	–	5.65×10^{1}	–	2.21×10^{1}
[Crashing]	10	6.44	4.49×10^{-5}	1.81×10^{-5}	5.02×10^{-5}	9.46×10^{-3}	8×10^{-6}
	10^2	24.61	6.03×10^{-3}	7.05×10^{-4}	6.80×10^{-4}	5.95×10^{-2}	1.25×10^{-4}
	10^4	46.90	2.50×10^{2}	2.85	1.50×10^{-1}	–	4.93×10^{-2}
	10^6	88.30	–	–	6.02×10^{1}	–	2.35×10^{1}
[FuelOpt]	10	2.93	8.46×10^{-5}	3.17×10^{-5}	6.62×10^{-5}	8.74×10^{-3}	2.20×10^{-5}
	10^2	5.31	1.22×10^{-2}	1.28×10^{-3}	7.98×10^{-4}	1.99×10^{-2}	4.21×10^{-4}
	10^4	9.53	2.43×10^{2}	4.81	1.95×10^{-1}	–	1.02×10^{-1}
	10^6	12.80	–	–	8.54×10^{1}	–	2.99×10^{1}

Results $m = n$

- Experiments with $m = n$

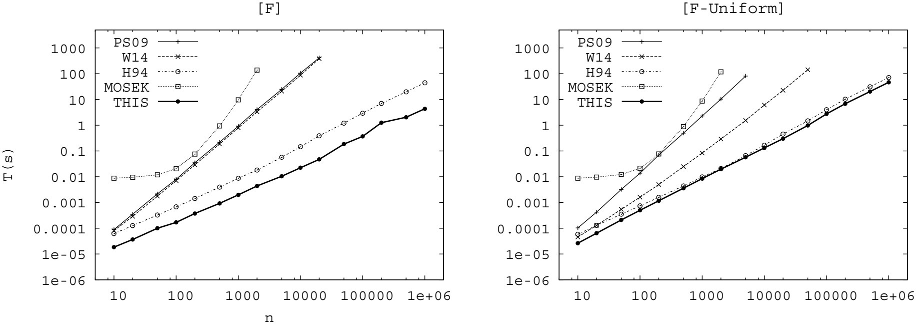


Figure : CPU Time(s) as a function of $n \in \{10, \ldots, 10^6\}$. $m = n$. Logarithmic representation

Results $m = n$

- Experiments with $m = n$

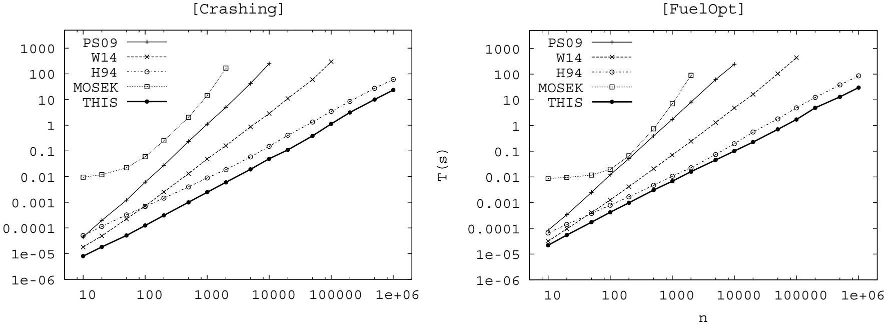


Figure : CPU Time(s) as a function of $n \in \{10, \ldots, 10^6\}$. $m = n$. Logarithmic representation

Results $m < n$

- Experiments with varying values of m, $m < n$.

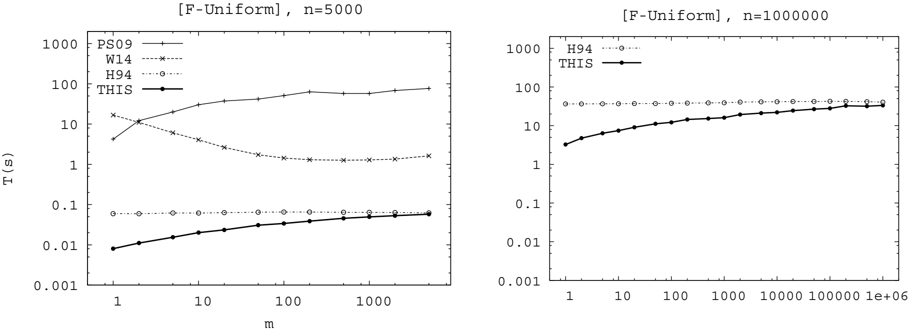


Figure : CPU Time(s) as a function of m. $n \in \{5000, 1000000\}$. Logarithmic representation

Results $m < n$

- Experiments with varying values of m, $m < n$.

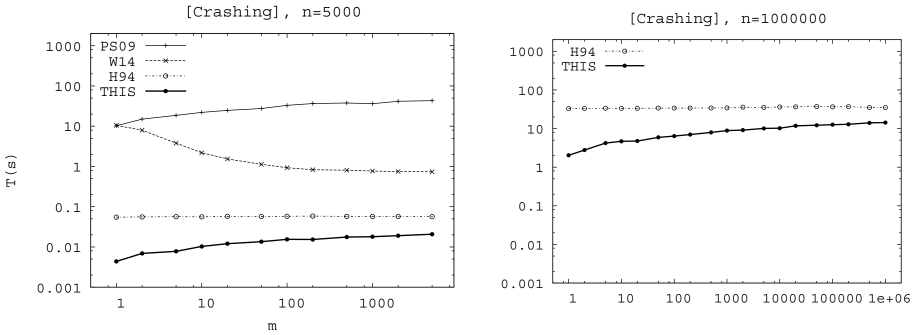


Figure : CPU Time(s) as a function of m. $n \in \{5000, 1000000\}$. Logarithmic representation

Results $m < n$

- Experiments with varying values of m, $m < n$.

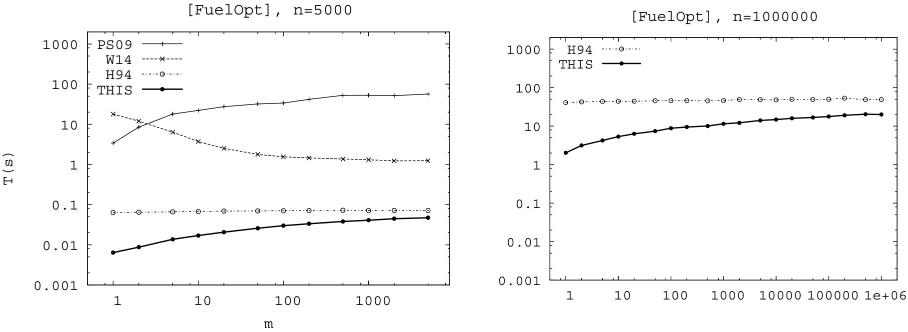


Figure : CPU Time(s) as a function of m. $n \in \{5000, 1000000\}$. Logarithmic representation

Conclusions

- Investigate a particular case of timing problem with flexible travel times, equivalent to a nested resource allocation problem.
- Highlighted a rich variety of applications
- Interesting geometrical properties
- A new polynomial algorithm
  - matching the state-of-the-art complexity (Hochbaum, 1994) when $m = n$
  - and improving when $\log m = o(\log n)$
- Different concepts based on monotonicity properties
- Extensive experimental analyses

Perspectives

- Resolution of series of problems with different permutations of activities
- Identifying an even richer set of related problems, models and applications
- Further generalizations

Thank you

THANK YOU FOR YOUR ATTENTION !

- For further reading:
  - T. Vidal, T. G. Crainic, M. Gendreau, and C. Prins. *A Unifying View on Timing Problems and Algorithms*. Submitted & revised to Networks. Tech. Rep. CIRRELT 2011-43.
  - T. Vidal, P. Jaillet, and N. Maculan, *A decomposition algorithm for nested resource allocation problems*. 2014. arXiv:1404.6694v1.
  - `http://w1.cirrelt.ca/~vidalt/`

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