Timing Problems and Vehicle Routing

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Context of this research

- General effort dedicated to better address *rich vehicle routing problems* involving many side constraints and *attributes*.

- Observation : several VRP settings deserve their *richness* to the temporal features they involve : Time windows, time-dependent cost and travel times, flexible travel times, stochastic travel times, break scheduling…

- The same questions are encountered in different domains: vehicle routing, scheduling, PERT, and isotone regression in statistics, among others.

- Leading us to a cross-domain analysis and classification of *timing problems and algorithms.*

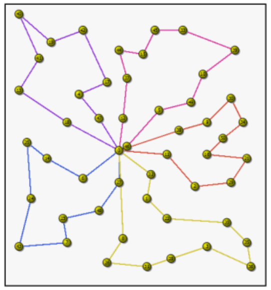
Presentation Outline

- Several applications presenting similar *timing* issues
- Timing features and problems
  - Classification and notation
  - Reductions
- A timing feature example: soft time-windows
- Timing Re-optimization

Several problems

- Four problems originating from different domains

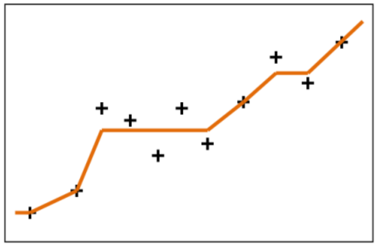
VRPTW



E/T scheduling

ship speed opt.

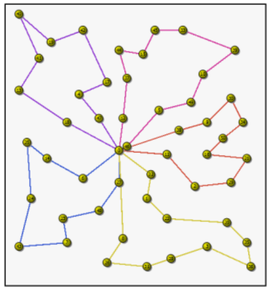
isotonic regression



Several problems

- Four problems originating from different domains:

VRPTW



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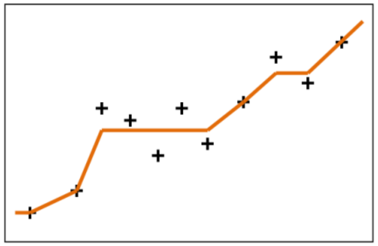


When visit sequence is fixed, optimizing on visit dates:

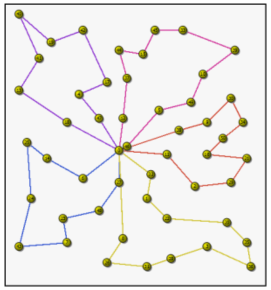
$$\min_{(t_1,\dots,t_n)\in\Re^{n+}} \sum_{i=1}^{n} \{\alpha(\bar{e}_i - t_i)^+ + \beta(t_i - \bar{l}_i)^+\}$$

$$\begin{aligned} s.t. \quad & t_i + p_i + d_{i,i+1} \leq t_{i+1} && 1 \leq i < n \\ & e_i \leq t_i \leq l_i && 1 \leq i \leq n \end{aligned}$$

Several problems

- Four problems originating from different domains:

VRPTW



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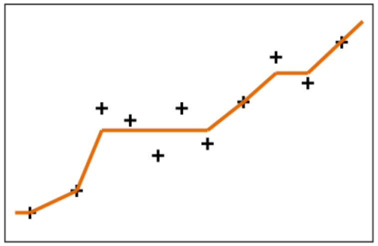


When visit sequence is fixed, optimizing on task execution dates:

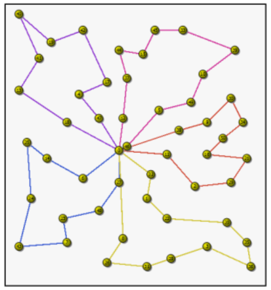
$$\min_{(t_1,\dots,t_n)\in\Re^{n+}} \sum_{i=1}^{n}\{\epsilon_i(d_i-t_i)^+ + \tau_i(t_i-d_i)^+\}$$

$$s.t. \quad t_i + p_i \le t_{i+1} \qquad 1 \le i < n$$

Several problems

- ❑ Four problems originating from different domains:

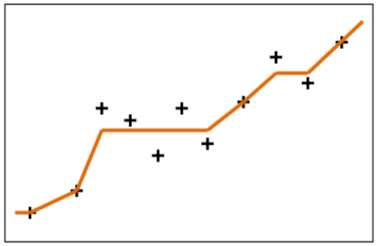
VRPTW



E/T scheduling

ship speed opt.

isotonic regression



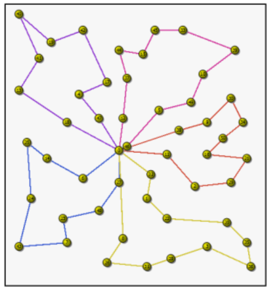
When visit sequence is fixed, fuel consumption optimization:

$$
\begin{aligned}
\min_{(t_1,\dots,t_n)\in\Re^{n+}} \quad & \sum_{i=1}^{n} d_{i,i+1}\,\hat{c}\left(\frac{d_{i,i+1}}{t_{i+1}-t_i}\right) & \\
\text{s.t.} \quad & t_i + p_i + d_{i,i+1}/v_{max} \leq t_{i+1} & 1 \leq i < n \\
& e_i \leq t_i \leq l_i & 1 \leq i \leq n
\end{aligned}
$$

Several problems

- Four problems originating from different domains:

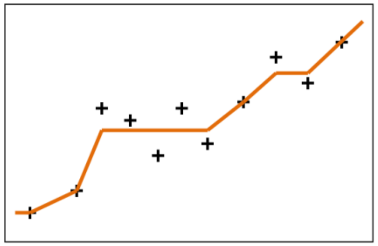
VRPTW

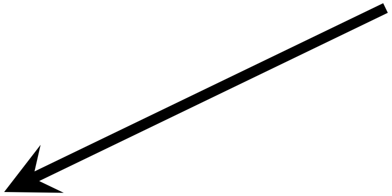


E/T scheduling

ship speed opt.

isotonic regression





$$\min_{\mathbf{t}=(t_1,\ldots,t_n)} \quad ||\mathbf{t} - \mathbf{N}||$$

$$t_i \leq t_{i+1} \qquad 1 \leq i < n$$

… with some characteristics in common

VRPTW

$$\begin{array}{lll} \displaystyle\min_{(t_1,\dots,t_n)\in\Re^{n+}} & \displaystyle\sum_{i=1}^{n}\{\alpha(\bar{e}_i - t_i)^+ + \beta(t_i - \bar{l}_i)^+\} & \\ s.t. & t_i + p_i + d_{i,i+1} \le t_{i+1} & 1 \le i < n \\ & e_i \le t_i \le l_i & 1 \le i \le n \end{array}$$

E/T scheduling

$$\begin{array}{lll} \displaystyle\min_{(t_1,\dots,t_n)\in\Re^{n+}} & \displaystyle\sum_{i=1}^{n}\{\epsilon_i(d_i - t_i)^+ + \tau_i(t_i - d_i)^+\} & \\ s.t. & t_i + p_i \le t_{i+1} & 1 \le i < n \end{array}$$

Isotonic regression

$$\begin{array}{lll} \displaystyle\min_{\mathbf{t}=(t_1,\dots,t_n)} & \|\mathbf{t} - \mathbf{N}\| & \\ & t_i \le t_{i+1} & 1 \le i < n \end{array}$$

Ship speed opt.

$$\begin{array}{lll} \displaystyle\min_{(t_1,\dots,t_n)\in\Re^{n+}} & \displaystyle\sum_{i=1}^{n} d_{i,i+1}\hat{c}\left(\frac{d_{i,i+1}}{t_{i+1}-t_i}\right) & \\ \text{s.t.} & t_i + p_i + d_{i,i+1}/v_{max} \le t_{i+1} & 1 \le i < n \\ & e_i \le t_i \le l_i & 1 \le i \le n \end{array}$$

TIMING

$$\begin{array}{lll} \displaystyle\min_{\mathbf{t}=(t_1,\dots,t_n)\in\Re^{n+}} & \displaystyle\sum_{F^x\in\mathcal{F}^{\text{OBJ}}} \alpha_x \sum_{1\le y\le m_x} f_y^x(\mathbf{t}) & \\ s.t. & t_i + p_i \le t_{i+1} & 1 \le i < n \\ & f_y^x(\mathbf{t}) \le 0 & F^x \in \mathcal{F}^{\text{CONS}},\ 1 \le y \le m_x \end{array}$$

Timing problems

TIMING

$$
\begin{aligned}
\min_{\mathbf{t}=(t_1,\dots,t_n)\in\Re^{n+}} \quad & \sum_{F^x\in\mathcal{F}^{\mathrm{OBJ}}} \alpha_x \sum_{1\leq y\leq m_x} f_y^x(\mathbf{t}) & \\
s.t. \quad & t_i + p_i \leq t_{i+1} & 1 \leq i < n \\
& f_y^x(\mathbf{t}) \leq 0 & F^x \in \mathcal{F}^{\mathrm{CONS}}\ ,\ 1 \leq y \leq m_x
\end{aligned}
$$

- ❑ Timing problems seek to determine the execution dates $(t_1,\dots,t_n)$ for a fixed sequence of activities.
- ❑ Totally ordered continuous variables
- ❑ Additional *features* F^x characterized by functions f_y^x for $1 \leq y \leq m_x$ that participate either in the objective or as constraints:
  - ➢ time windows, time-dependent proc. times, flexible travel times, time lags, no waiting, limited waiting, and so on…

Timing problems

TIMING

$$
\begin{array}{lll}
\displaystyle\min_{\mathbf{t}=(t_1,\dots,t_n)\in\Re^{n+}} & \displaystyle\sum_{F^x\in\mathcal{F}^{\mathrm{OBJ}}} \alpha_x \sum_{1\leq y\leq m_x} f_y^x(\mathbf{t}) & \\
s.t. & t_i + p_i \leq t_{i+1} & 1 \leq i < n \\
 & f_y^x(\mathbf{t}) \leq 0 & F^x \in \mathcal{F}^{\mathrm{CONS}} \;,\; 1 \leq y \leq m_x
\end{array}
$$

- Several names in the literature: *Scheduling*, *Timing*, *Projections onto Order Simplexes, Optimal service time problem …*
- Few dedicated studies, literature scattered among several research domains despite its relevance to many applications
- Thus motivating a dedicated review and analysis of timing algorithms to fill the gap.

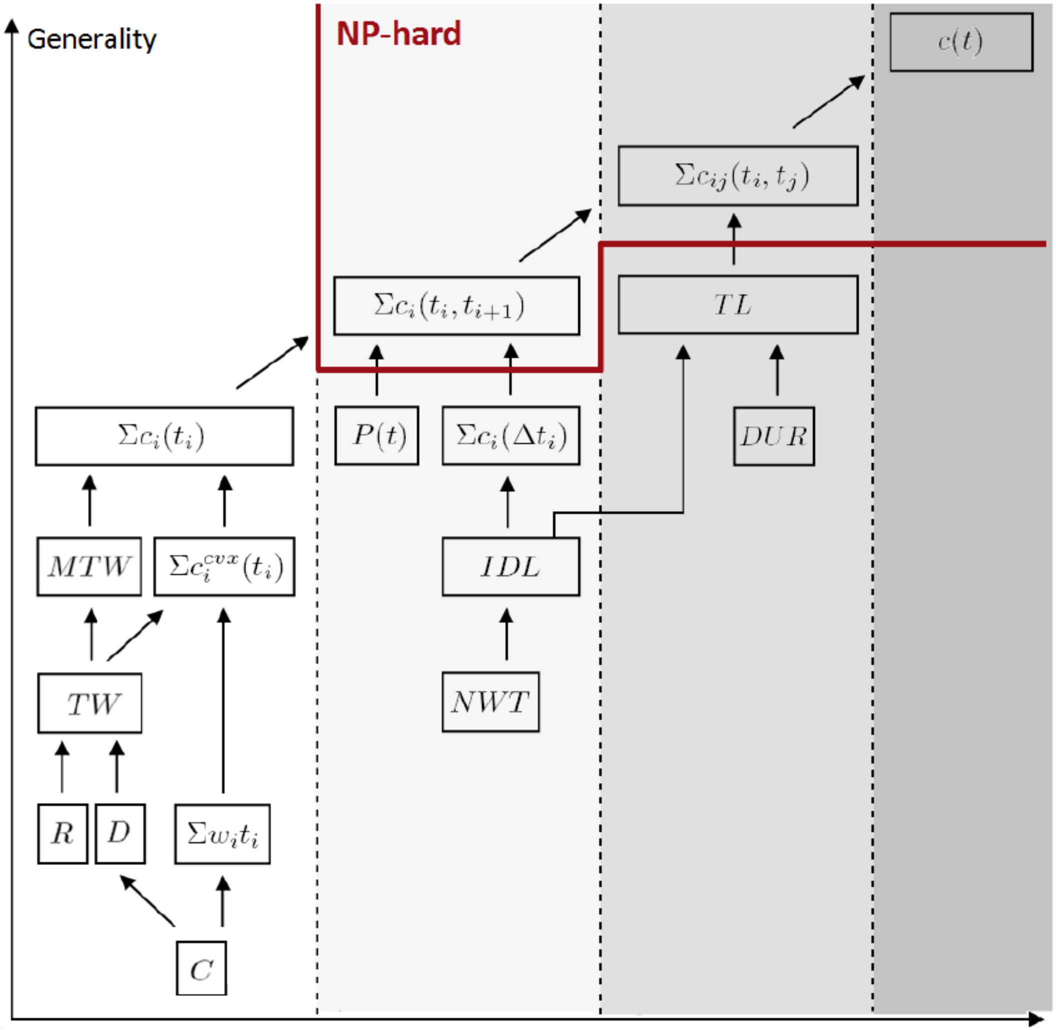
Timing features from the vehicle routing domain

- **Rich vehicle routing problems** can involve various *timing features*

Symbol	Parameters	Char. functions	Most frequent roles
D	due dates d_i	$f_i(\mathbf{t}) = (t_i - d_i)^+$	Service deadlines constraints, tardiness
R	release dates r_i	$f_i(\mathbf{t}) = (r_i - t_i)^+$	Release-dates, earliness.
TW	time windows $TW_i = [e_i, l_i]$	$f_i(\mathbf{t}) = (t_i - l_i)^+ + (e_i - t_i)^+$	Time-window constraints, soft time windows.
MTW	multiple TW $MTW_i = \cup[e_{ik}, l_{ik}]$	$f_i(\mathbf{t}) = \min_k [(t_i - l_{ik})^+ + (e_{ik} - t_i)^+]$	Multiple time-window constraints
$\Sigma c_i(t_i)$	general $c_i(t)$	$f_i(\mathbf{t}) = c_i(t_i)$	Time-dependent service costs
$\Sigma c_i^{\text{cvx}}(t_i)$	convex $c_i^{\text{cvx}}(t_i)$	$f_i(\mathbf{t}) = c_i^{\text{cvx}}(t_i)$	Time-d. convex service costs
DUR	total dur. δ_{max}	$f(\mathbf{t}) = (t_n - \delta_{max} - t_1)^+$	Duration or overall idle time
NWT	no wait	$f_i(\mathbf{t}) = (t_{i+1} - p_i - t_i)^+$	No wait constraints
IDL	idle time ι_i	$f_i(\mathbf{t}) = (t_{i+1} - p_i - \iota_i - t_i)^+$	Limited idle time per stop, min idle time excess
$P(t)$	time-dependent proc. times $p_i(t_i)$	$f_i(\mathbf{t}) = (t_i + p_i(t_i) - t_{i+1})^+$	Time-dependent driving-times
TL	time-lags δ_{ij}	$f_i(\mathbf{t}) = (t_j - \delta_{ij} - t_i)^+$	Time-lag constraints
$\Sigma c_i(\Delta t_i)$	general $c_i(t)$	$f_i(\mathbf{t}) = c_i(t_{i+1} - t_i)$	Flexible travel times
$\Sigma c_{ij}(t_i, t_j)$	general $c_{ij}(t, t')$	$f_{ij}(\mathbf{t}) = c_i(t_i, t_j)$	Separable objectives or constraints by any pairs of variables ...

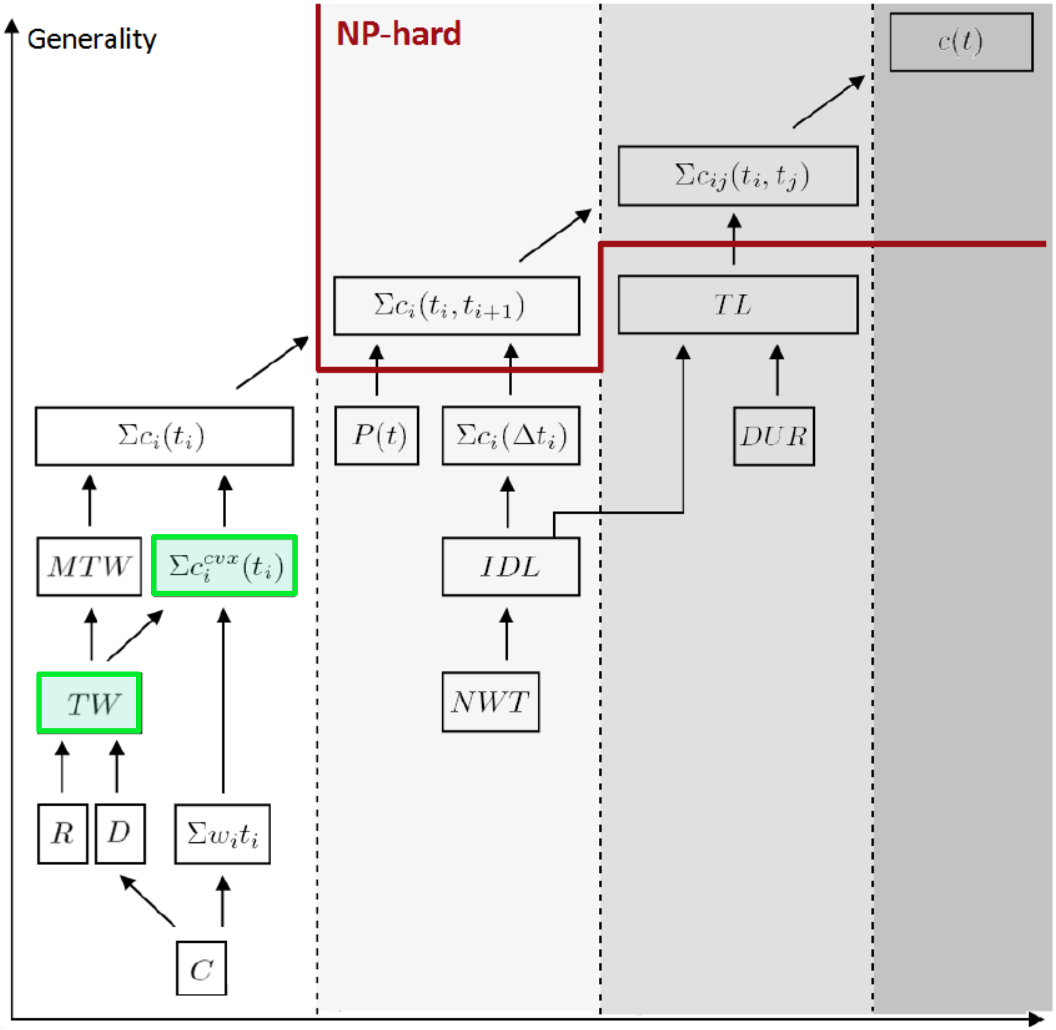
Timing features hierarchy

- ❑ These features can be classified and hierarchized (many-one linear reduction relationships between the associated timing problems)
- ❑ Features in the NP-hard area lead to NP-hard timing problems



Timing features hierarchy

- ❑ In this presentation, brief glimpse of the analysis.
- ❑ We examine a particular feature as illustrative example →
- ❑ A similar study has been conducted on other features from this figure.



A feature example: soft time-windows

- ☐ Timing problem
  - with soft time-windows (penalized early and late arrival)
  - and generally with any convex separable cost

$$\begin{aligned}
\min_{(t_1,\dots,t_n)\in\Re^{n+}} \quad & \sum_{i=1}^{n} \{\alpha(\bar{e}_i - t_i)^+ + \beta(t_i - \bar{l}_i)^+\} \\
s.t. \quad & t_i + p_i \leq t_{i+1} \qquad 1 \leq i < n
\end{aligned}$$

$$\begin{aligned}
\min_{(t_1,\dots,t_n)\in\Re^{n}} \quad & \sum_{i=1}^{n} c_i^{\mathrm{CVX}}(t_i) \\
s.t. \quad & t_i + p_i \leq t_{i+1}
\end{aligned}$$

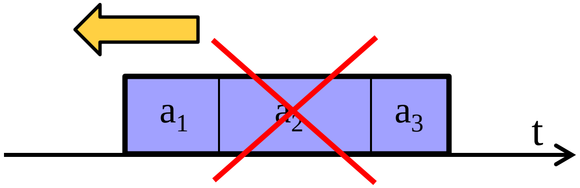
- ☐ We inventoried more than 30 algorithms from various domains (routing, scheduling, PERT, statistics…) that address these models.

- ☐ The solution block representation / active set framework (Chakravarti 1989, Best & Chakravarti 1990, Best et al. 2000, Ahuja & Orlin 2001) can be used to characterize these methods. But we need to generalize the optimality conditions to the non-smooth case.

A feature example: soft time-windows

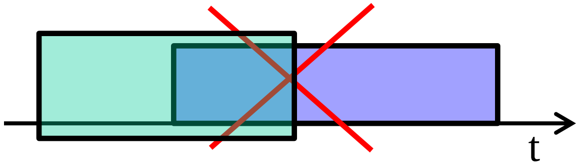
- A block B is defined as a subsequence of activities $(a_{B(1)},\dots,a_{B(|B|)})$ processed consecutively (such that $t_i + p_i = t_{i+1}$)

- Theorem: Let costs $c_i(t_i)$ be proper convex, eventually non-smooth, functions. **A solution $(t^*_1,\dots,t^*_n)$ of the timing problem with convex separable costs is optimal if and only if** it can be assimilated to a succession of activity blocks $(B_1,\dots,B_m)$ such that:



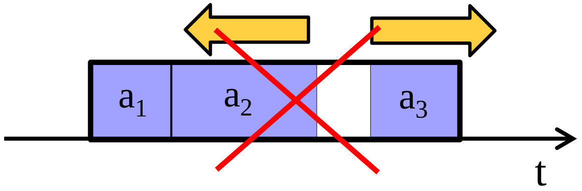
1) **Blocks are optimally placed**: for each block B_i,

$$t^*_{B_i(1)} \in \mathit{argmin}\ C_{B_i}(t)$$



2) **Blocks are spaced**: for each pair of blocks (B_i,B_{i+1}),

$$t^*_{B_i(1)} + \Sigma p_{B_i(j)} < t^*_{B_{i+1}(1)}$$



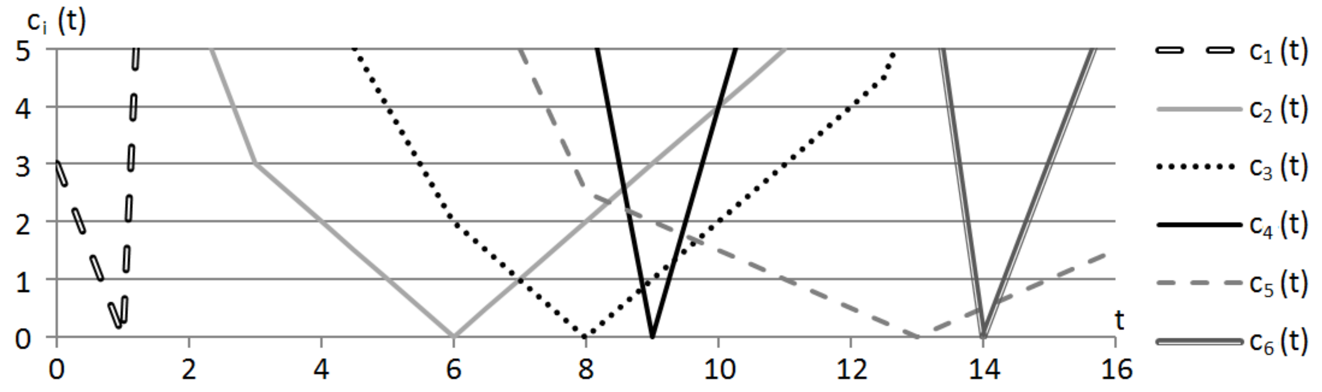
3) **Blocks are consistent**: for each block B_i and prefix block B_i^k,

$$\mathit{max\ argmin}\ C_{B_i}{}^k(t) \geq t^*_{B_i(1)}$$

A feature example: soft time-windows

- Three main families of algorithms can be identified:
  - Primal feasible, that respect *spacing condition 2*
  - Dual feasible, that respect *consistency condition 3*
  - Dynamic programming

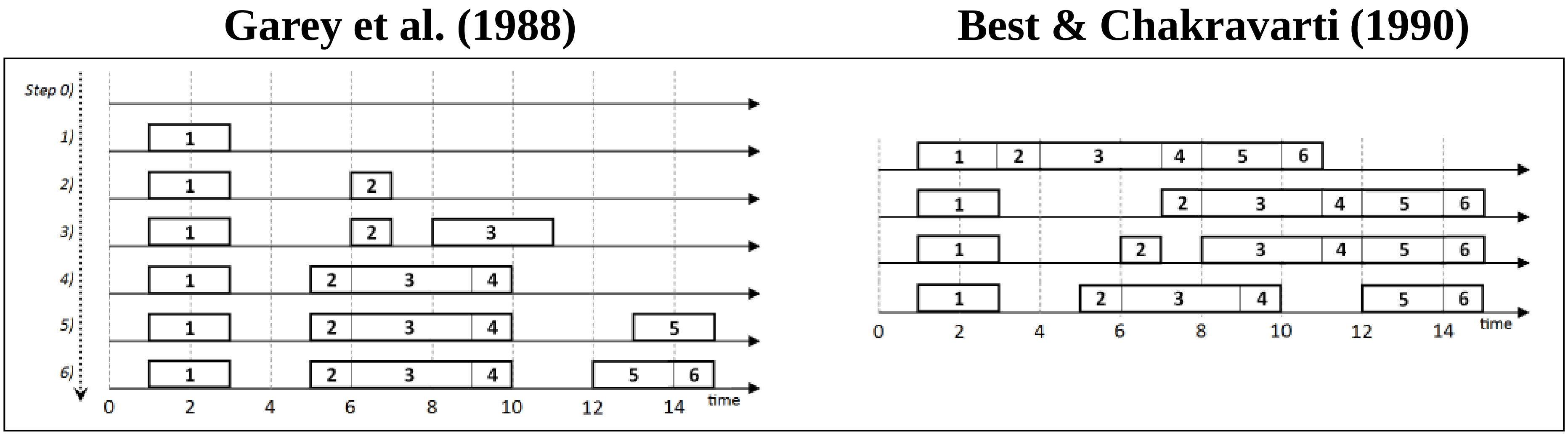
- To illustrate, consider this small problem with 6 activities



Act. i	p_i
1	2
2	1
3	3
4	1
5	2
6	1

A feature example: soft time-windows

- **Primal feasible method, respecting the spacing condition.**
  - Brunk (1955) : Minimum Lower Set Algorithm in O(n²) unimodal minimizations.
  - Extended by Garey et al. (1988) and Best & Chakravarti (1990) to work, respectively, in O(n log n) elementary operations in the case of (E/T) scheduling, and O(n) unimodal function minimizations in the general convex case.

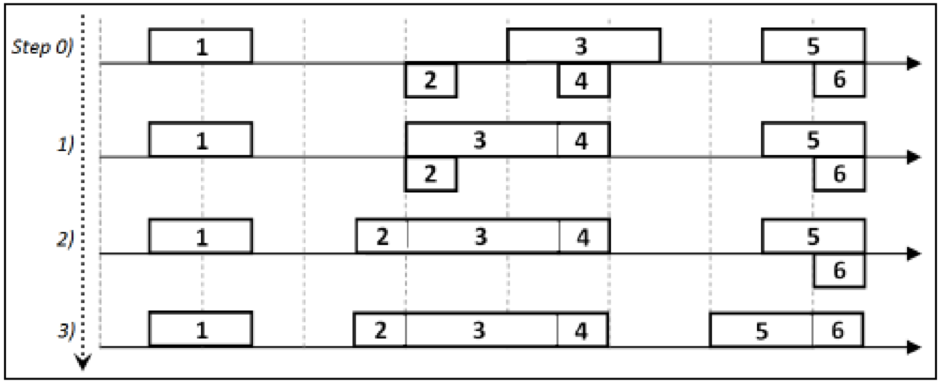

Garey et al. (1988)

Best & Chakravarti (1990)

- Other related methods originating from the field of (E/T) scheduling: Davis and Kanet 1993, Wan and Yen 2002…
- In the context of PERT with convex costs : Chrétienne and Sourd (2003)

A feature example: soft time-windows

- **Dual feasible method, respecting the consistency condition.**
  - Ayer et al. (1955) : Pool Adjacent Violator Algorithm (PAV).



  - Extended to the general convex case by Best et al. (2000) & Ahuja and Orlin (2001) -> O(n) unimodal function minimizations
  - Can work in $O(n \log^2 n)$ for Isotone Regression with $|| \, ||_1$ (equivalent to (E/T) with equal penalties for earliness and tardiness) (Pardalos 1995)
  - For the VRP with convex service costs, Dumas et al. (1990) can be viewed as another application of this principle

A feature example: soft time-windows

- **Dynamic programming-based methods** (Yano and Kim 1991, Sourd 2005, Ibaraki et al. 2005, 2008, Hendel and Sourd 2007, Hashimoto et al. 2006, 2008)

- Forward dynamic programming

$$F_i(t) = \min_{0 \leq x \leq t} \{c_i(x) + F_{i-1}(x - p_{i-1})\}$$

- Backward dynamic programming

$$B_i(t) = \min_{x \geq t} \{c_i(x) + B_{i+1}(x + p_i)\}$$

Timing problems

- Hence, many different methods for this particular feature example. The literature on timing problems is rich, but scattered. All in all, 26 different methods from different domains were classified as variations of 3 main algorithmic ideas.

Timing re-optimization

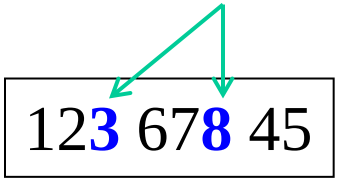
- ❑ Furthermore, when used within LS, solving all timing problems *from scratch* is generally not efficient

- ❑ The general goal when exploring neighborhoods is to solve N successive timing problems with different activity permutations σ^k.

$$\begin{aligned} \min_{\mathbf{t}=(t_1,\dots,t_n)\in\Re^{n+}} \quad & \sum_{F^x \in \mathcal{F}^{\mathrm{OBJ}}} \alpha_x \sum_{1 \leq y \leq m_x} f_y^x(\mathbf{t}) \\ s.t. \quad & t_{\sigma^k(i)} + p_{\sigma^k(i)\sigma^k(i+1)} \leq t_{\sigma^k(i+1)} \\ & f_y^x(\mathbf{t}) \leq 0 \end{aligned}$$

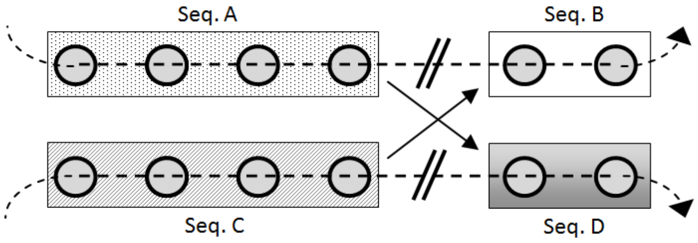
Timing re-optimization

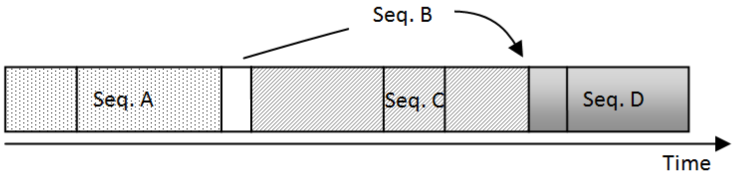
- In classical VRP neighborhoods, the neighborhood size is often rather large: $|N| = \Omega(n^2)$, and permutations are very particular.
  - They have a bounded number (often <= 4) of *breakpoints: integers x such that* $\sigma(x)+1 \neq \sigma(x+1)$,



12**3** 67**8** 45

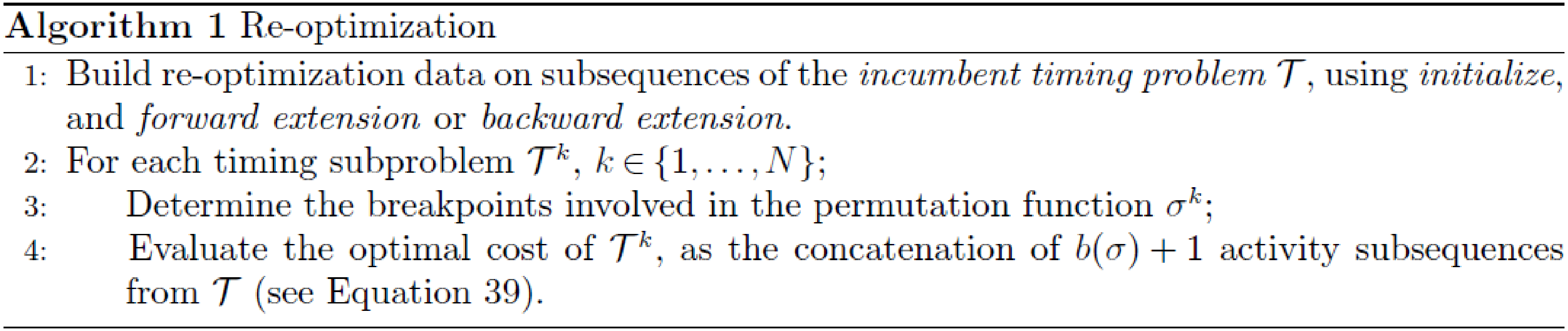
  - The resulting sequences of activities can be assimilated to recombinations of a bounded number of subsequences.





Timing re-optimization

- ❑ Management of information of subsequences, efficient *timing re-optimization* by means of a subset of 4 procedures, used within local searches:
  - ➢ Initialization of suitable re-optimization data for a single activity
  - ➢ Forward (F) or backward (B) computation of data on larger subsequences
  - ➢ Evaluation of a concatenation of two (C2) or more (C3+) subsequences



Algorithm 1 Re-optimization

```
Build re-optimization data on subsequences of the incumbent timing problem $\mathcal{T}$, using initialize, and forward extension or backward extension.
For each timing subproblem $\mathcal{T}^k$, $k \in \{1, \dots, N\}$;
    Determine the breakpoints involved in the permutation function $\sigma^k$;
    Evaluate the optimal cost of $\mathcal{T}^k$, as the concatenation of $b(\sigma) + 1$ activity subsequences from $\mathcal{T}$ (see Equation 39).
```

Timing re-optimization

- Example of soft time-windows: Forward and backward extension to compute data on subsequences, and *evaluate concatenation* of 2 sequences (Ibaraki et al. 2005, 2008):

$$Z^*(A_1 \oplus A_2) = \min_{t \geq 0}\{F(A_1)(t) + B(A_2)(t + p_{A_1(|A_1|)A_2(1)})\}$$

  - In the convex case, the concatenation of 3+ sequences is also addressed efficiently.
  - O(log ϕ) for convex piecewise functions with a total of ϕ pieces.
  - O(log n) move evaluations for soft TW

Conclusions of this analysis

- For other features: Surveying the literature, we classified many re-optimization based methodologies from various domains, and for a large variety of attributes. (Savelsbergh 1985,1992, Kindervater and Savelsbergh 1997, Campbell and Savelsbergh 2004, Ergun and Orlin 2006, Irnich 2008, Hashimoto et al. 2006,2008, Kedad-Sidhoum and Sourd 2010)…

- We could identify a set of state-of-the-art timing methods, which are the key to solve many rich VRP settings:

Conclusions of this analysis

Problem	From Scratch		Re-opt. by concat.	F/B	C2	C3+	Sd	Assumptions
$\{W\|\varnothing\}$	Min idle time	$O(n)$	—	$O(1)$		$O(1)$	✓	
$\{\varnothing\|TW\}$	Min idle time	$O(n)$	Savelsbergh (1985) & Kind. and Sav. (1997)	$O(1)$		$O(1)$	✓	
$\{D\|\varnothing\}$	Min idle time	$O(n)$	Ergun and Orlin (2006)	$O(\log n)$	$O(1)^*$	—		penalty coefficient depending upon act.
$\{D, R(d_i = r_i)\|NWT\}$	Min idle time	$O(n)$	Kedad-Sidhoum and Sourd (2010)	$O(\log n)$	$O(1)^*$	—		penalty coefficient depending upon act.
$\{D, R(d_i = r_i)\|\varnothing\}$	Garey et al. (1988) & Ahuja and Orlin (2001)	$O(n \log n)$	Ibaraki et al. (2008)	$O(\log n)$		$O(\log n)$	✓	
$\{D\|R\}$	Min idle time	$O(n)$	Ibaraki et al. (2008)	$O(\log n)$		$O(\log n)$	✓	
$\{\Sigma c_i^{\text{cvx}}(t_i)\|\varnothing\}$	Ibaraki et al. (2008)	$O(n \log \varphi_c)$	Ibaraki et al. (2008)	$O(\log \varphi_c)$		$O(\log \varphi_c)$	✓	cost f. ≥ 0, p.l. & l.s.c
$\{\Sigma c_i(t_i)\|\varnothing\}$	Ibaraki et al. (2005)	$O(n\varphi_c)$	Ibaraki et al. (2005)	$O(\varphi_c)$		$O(\varphi_c)$	✓	cost f. ≥ 0, p.l. & l.s.c
$\{\varnothing\|MTW\}$	Min idle time	$O(n + \varphi_{\text{MTW}})$	Ibaraki et al. (2005)	$O(\log \varphi_{\text{MTW}})$		—	✓	
$\{DUR\|TW\}$, $\{\varnothing\|DUR, TW\}$	Malcolm et al. (1959)	$O(n)$	Savelsbergh (1992) & Kind. and Sav. (1997)	$O(1)$		$O(1)$	✓	
$\{DUR\|MTW\}$, $\{\varnothing\|DUR, MTW\}$	Tricoire et al. (2010)	$O(n\varphi_{MTW})$	Hashimoto et al. (2006)	$O(\varphi_{\text{MTW}})$		—	✓	
$\{\varnothing\|IDL, TW\}$	Hunsaker and S. (2002)	$O(n)$	—	—	—	—		
$\{\Sigma c_i^{\text{cvx}}(\Delta t_i), \Sigma c_i(t_i)\|\varnothing\}$	Sourd (2005) & Hashimoto et al. (2006)	$O(n(\varphi_c + \widehat{\varphi_c} \times \varphi'_c))$	Sourd (2005) & Hashimoto et al. (2006)	$O(\varphi_c + \widehat{\varphi_c} \times \varphi'_c)$		—	✓	cost f. ≥ 0, p.l. & l.s.c
$\{D\|R, P(t)\}$	Min idle time	$O(n)$	—	—	—	—		FIFO assumption
$\{\varnothing\|TW, P(t)\}$	Donati et al. (2008)	$O(n)$	Donati et al. (2008)	$O(1)$		—	✓	FIFO assumption
$\{\Sigma c_i(t_i)\|P(t)\}$	Hashimoto et al. (2008)	$O(n(\varphi_c + \varphi_p))$	Hashimoto et al. (2008)	$O(\varphi_c + \varphi_p)$		—	✓	cost f. ≥ 0, p.l. & l.s.c & HYI assumption
$\{\varnothing\|TL, TW\}$	Hurink and Keuchel (2001)	$O(n^3)$	—	—	—	—		
$\{\varnothing\|TL, TW\}$	Haugland and Ho (2010)	$O(n \log n)$	—	—	—	—		$O(n)$ TL constraints
$\{DUR > D > TL\|R\}$	Cordeau and Laporte (2003)	$O(n^2)$	—	—	—	—		$O(n)$ TL constraints & LIFO assumption
$\{\Sigma c_{ij}^{\text{cvx}}(t_j - t_i), \Sigma c_i^{\text{cvx}}(t_i)\|\varnothing\}$	Ahuja et al. (2003)	$O(n^3 \log n \log(nU))$	—	—	—	—		U is an upper bound of execution dates

Conclusions of this analysis

- ❑ Large analysis of a rich body of problems with time characteristics and totally ordered variables. Cross-domain synthesis, considering methods from various fields such as vehicle routing, scheduling, PERT, and isotonic regression. Identification of main resolution principles

- ❑ For several "rich" combinatorial optimization settings, the timing sub-problems represent the core of "richness" and deserve particular attention.

- ❑ Furthermore, timing sub-problems frequently arise in the context of local search, and thus we analyzed both stand-alone resolution and efficient solving of series of problems.

Perspectives

- Timing procedures have been integrated in a recent Unified Hybrid Genetic Search
- Several features and feature combinations were identified in this work, for which new timing algorithms (including re-optimization procedures) should be sought.
- Generalization to other cumulative resources, multi-objective or stochastic settings.
- Further studies on complexity lower bounds.

- For further reading on timing problems and algorithms: Vidal T., Crainic T.G., Gendreau M., Prins C. A Unifying view on Timing Problems and Algorithms (2011), CIRRELT Tech.Rep. 2011-43.

- For a survey on vehicle routing variants with time attributes: Vidal T., Crainic T.G., Gendreau M., Prins C. Heuristics for Multi-Attribute Vehicle Routing Problems: A Survey and Synthesis. (2012), CIRRELT Tech. Rep. 2012-05.

- **Thank you very much for your attention**

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