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# The Minimum Distance Superset Problem: Formulations and Algorithms

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**Abstract.** The partial digest problem consists in retrieving the positions of a set of points on the real line from their unlabeled pairwise distances. This problem is critical for DNA sequencing, as well as for phase retrieval in X-ray crystallography. When some of the distances are missing, this problem generalizes into a "minimum distance superset problem", which aims to find a set of points of minimum cardinality such that the multiset of their pairwise distances is a superset of the input.

We introduce a quadratic integer programming formulation for the minimum distance superset problem with a pseudo-polynomial number of variables, as well as a polynomial-size integer programming formulation. We investigate three types of solution approaches based on an available integer programming solver: 1) solving a linearization of the pseudo-polynomial-sized formulation, 2) solving the complete polynomial-sized formulation, or 3) performing a binary search over the number of points and solving a simpler feasibility or optimization problem at each step. As illustrated by our computational experiments, the polynomial formulation with binary search leads to the most promising results, allowing to optimally solve most instances with up to 25 distance values and 8 solution points.

**Keywords.** Partial digest problem, Minimum distance superset, Turnpike problem, Combinatorial optimization, Integer programming

## 1 Introduction

Computing all pairwise distances from a set of $n$ points on the real line is simple. In contrast, from an unlabeled set of $\binom{n}{2}$ pairwise distances, retrieving the positions of a set of $n$ points on the line is a much greater challenge, which has attracted a significant research effort in recent years, as reviewed in details in Lemke et al. (2003). This problem is commonly known as the partial digest problem (PDP – Skiena and Sundaram 1994), and sometimes called turnpike problem (Dakic, 2000), or chords' problem (Daurat et al., 2002, 2005).

Remarkably, whether this problem can be solved in polynomial time remains a long-standing open research question. An efficient backtracking algorithm, due to Skiena et al. (1990), achieves a computational complexity of $O(2^n n \log n)$ in the worst case and $O(n^2 \log n)$ with probability 1 for a random problem instance. Zhang (1994) shows an instance on which the exponential behavior occurs. Interestingly, this particular instance also turns out to be solvable in polynomial time via a semi-definite programming algorithm due to Dakic (2000). If the points of the solution have to be placed in a $k$-dimensional space, then the problem is known to be NP-hard (Skiena et al., 1990).

The first known appearance of the PDP comes from phase retrieval in X-ray crystallography (Patterson, 1935, 1944). Later on, the problem became of major importance due to its applications for DNA sequencing (Skiena et al., 1990; Skiena and Sundaram, 1994). Indeed, a DNA molecule can be viewed as a string of nucleotides {A, C, G, T}. A restriction enzyme is a chemical that cuts DNA at specific sequence patterns of nucleotides, called restriction sites. A digestion experiment is performed by allowing a restriction enzyme to digest several clones of DNA molecule and measuring the lengths of the resulting fragments. Retrieving the original locations of the restriction sites from the lengths of the fragments becomes an instance of PDP.

Experiments and measurements are also never perfect. Four main sources of errors of digest experiments are discussed in (Cieliebak et al., 2005): additional fragments, missing fragments, measurement errors and multiplicity detection. These errors complicate the task of retrieving viable reconstructions of the molecule's restriction sites. In particular, Skiena et al. (1990) proved that the partial digestion problem is strongly NP-hard if additive error bounds are assigned to each distance individually, while Cieliebak et al. (2003) proved that the problem variant with missing fragments is NP-hard. Skiena and Sundaram (1994) generalized the backtracking algorithm to address small measurement errors and inaccurate multiplicity detection. These important variants of the PDP have been generally overlooked in the literature, and no efficient algorithm has been proposed to solve problems with an arbitrarily high number of missing fragments.

To fill this methodological gap, we investigate in this article the specific case of the PDP with missing fragments. In this context, the goal of the reconstruction is to find the smallest set of restriction sites locations which produces the input distances. We will refer to this problem variant as the *minimum distance superset problem* (MDSP). A key challenge of this problem relates to the high number of symmetrical feasible solutions, which cripples the performance of tree search-based algorithms. Not only there usually exists several solutions that are equivalent via translation and mirroring, but it is almost always possible to find similar solutions via permutations of distances.

To solve the MDSP, we propose a quadratic programming formulation with a pseudo-polynomial number of variables, which generalizes a quadratic PDP formulation from Dakic

(2000), as well as some variable reductions and a reformulation-linearization of the model. This approach is, however, only practicable when the number of distinct and non-negative linear combinations of the input distances with coefficients in $\{-1, 0, 1\}$ is relatively small. For the general (NP-hard) problem, we introduce a new integer programming formulation with a polynomial number of variables, as well as a decomposition technique for this formulation based on binary search on the number of solution points. Our experimental analysis show that the polynomial formulation can solve problems of realistic size for DNA sequencing applications, with up to 25 distances and eventual missing fragments, laying a first foundation for the future development of more advanced mathematical programming approaches.

## 2 Problem Statement

In the remainder of this paper, and unless specified, we will use the curly brackets $\{\}$ to refer to *multisets* (sets that allow possible repeated elements). Subtracting an element from a multiset removes a single repetition, for example: $\{1, 1, 1, 3, 9, 12\} - \{1, 1, 3\} = \{1, 9, 12\}$. Repeated elements are taken into account in the cardinality and inclusion operators, e.g., $\{1, 3\} \subset \{1, 1, 3, 4\}$ but $\{1, 1, 3\} \not\subset \{1, 3, 4\}$, and $|\{1, 1, 3\}| = 3$. Moreover, for a set $P$, we define as $\Delta(P)$ the multiset $\{q - p \,|\, p, q \in P, p < q\}$ of pairwise distances of a set $P$. The *minimum distance superset problem* (MDSP) can now be stated as follows:

**MDSP:** Given a multiset $D = \{d_1, d_2, \ldots, d_k\}$ of $k$ positive integers, find the smallest set $P \subset \mathbb{Z}$ such that $D \subseteq \Delta(P)$.

Denoting as $n$ the size of the set $P$, we know that $n \leq k + 1$, since it is always possible to reconstruct $k$ distances with $k + 1$ points. Moreover, $|D| \leq |\Delta(P)|$ and thus $k \leq \binom{n}{2}$, leading to the following trivial bounds:

$$1/2 + \sqrt{1/4 + 2k} \leq n \leq k + 1. \tag{1}$$

The MDSP generalizes the well-known partial digest problem (PDP), as an algorithm for the former can solve the latter. More exactly, the PDP aims to verify whether the trivial lower bound (and thus the equality $k = \binom{n}{2}$) is attained:

**PDP:** Given a multiset $D = \{d_1, \ldots, d_k\}$ of $k = \binom{n}{2}$ positive integers, is there a set $P = \{p_1, \ldots, p_n\}$ of $n$ points on a line such that $D = \Delta(P)$?

Note that both the PDP and MDSP can have more than one noncongruent solution set $P$. Bloom's distance set $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17\}$ (illustrated in Skiena et al. 1990) is a good example, as two distinct reconstructions, illustrated in Figure 1, are possible: $P_1 = \{0, 1, 4, 10, 12, 17\}$ and $P_2 = \{0, 1, 8, 11, 13, 17\}$ such that $\Delta(P_1) = \Delta(P_2)$. It

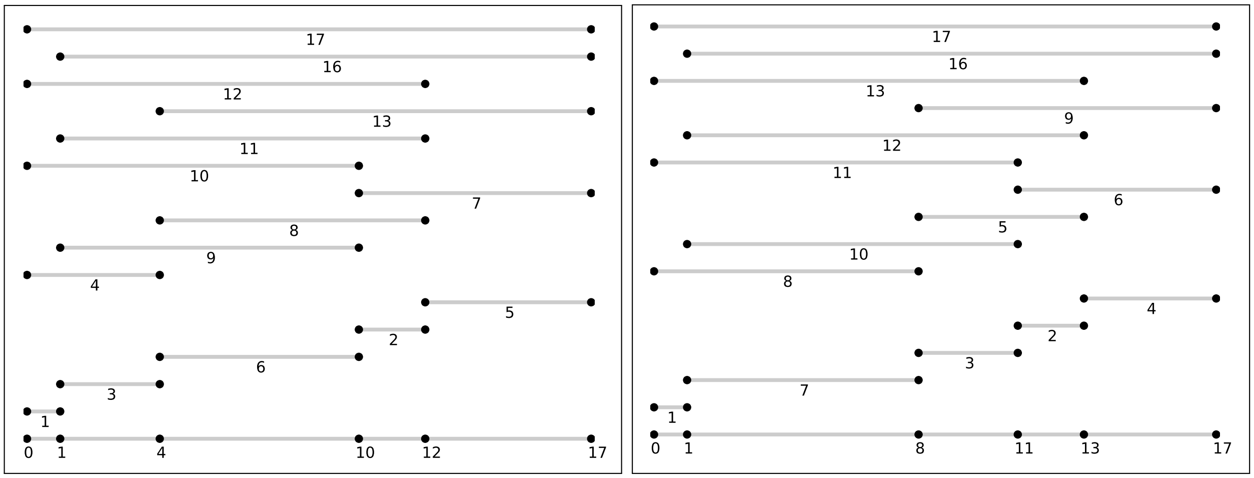



Figure 1: Two possible reconstructions for Bloom's distance set

is also important to notice that these two solutions are not *congruent*, as there is no set of rigid motions on the real line that can turn one solution into the other. For the PDP, Skiena et al. (1990) proved that the maximum number of homometric reconstructions of an instance of cardinality $k = n(n-1)/2$ has an upper bound of $0.5n^{1.12334827}$. In contrast, in the case of the MDSP, the number of non-congruent optimal reconstructions can grow exponentially. For example, for $D = \{1, 2, 4, \ldots, 2^{k-1}\}$, there are at least $k!$ optimal solutions of the form

$$P = \left\{ 0, \sum_{d \in S_1} d, \ldots, \sum_{d \in S_k} d \right\} \text{ with } S_1 \subset S_2 \subset \cdots \subset S_k \text{ and } S_k = D. \tag{2}$$

Such a high level of symmetry tends to complicate the resolution of the MDSP via branch-and-bound based techniques.

# 3 Quadratic Formulation

This section introduces a quadratically constrained quadratic programming (QCQP) formulation of the MDSP. QCQP is $\mathcal{NP}$-hard in its general form, and plays an important role as a modeling tool for various problems. Dakic (2000) used 0–1 QCQP to a large extent to solve the PDP, and the formulation proposed in this section is a natural extension of this model.

First, let $B$ be the sum of all distances in $D$. Without loss of generality, we can restrict our search to MDSP solutions with points in $\{0, \ldots, B\}$. Indeed, any optimal solution $s$ containing points with negative coordinates can be translated into an equivalent solution containing the point 0 as well as other points with non-negative coordinates. Moreover, if we suppose the existence of an optimal solution $s$ containing the point 0 along with a point $p > B$, then the set of points $P(s)$ can be split into two disjoint subsets, such that $P(s) = P_1 \cup P_2$, $P_1 \cap P_2 = \varnothing$, and $\Delta(P_1) \cup \Delta(P_2) = D$. As a consequence, it is possible to translate all the points in $P_2$ while maintaining feasibility, until one point in $P_2$ overlaps with one point of $P_1$. This effectively decreases the number of points in $s$ by one, and contradicts its optimality. Based on these observations, the MDSP can be formulated as:

$$\min \quad \sum_{i=0}^{B} x_i \tag{3}$$

$$\text{s.t.} \quad \sum_{i=0}^{B-d} x_i x_{i+d} \geq \text{mult}(d) \qquad \forall d \in D' \tag{4}$$

$$x_0 = 1 \tag{5}$$

$$x_i \in \{0, 1\} \qquad 0 \leq i \leq B \tag{6}$$

In this formulation, $D'$ is the set of unique distances in $D$, and $\text{mult}(d)$ is the multiplicity of a distance $d$. Each binary variable $x_p$ represents an integer $p$ between 0 and $B$, which is valued to 1 if and only if the point $p$ is used in the solution. Equation (3) seeks to minimize the number of points, while Equation (4) imposes the number of occurrences of each distance $d$, and Equation (5) finally ensures that the point 0 is used.

Despite its simplicity, this formulation has two essential drawbacks. First, the number of variables depends on the sum of the distances $B$, and thus the size of this formulation is pseudo-polynomial. Second, its constraints are not always convex, which is an issue for most available quadratic programming solvers.

## 3.1 Variables Filtering

The number of variables of the previous formulation can be reduced in two ways. The first way is to divide all integers in $D$ by their greatest common divisor. This method, used for all algorithms of this paper, is only effective when no pair of numbers in $D$ are relatively prime, as their greatest common divisor will be 1. The second one is to use the procedure shown in Algorithm 1. It calculates all possible linear combinations of input distances, with the added constraint that the only coefficients allowed are $-1$, 0 and 1. Then, it removes all negative elements and returns the resulting set of points. This method can lead to a substantial decrease in the size of the formulation, if the distances are sparse enough. However, the formulation remains pseudo-polynomial, since $\Theta(\min\{B, 3^k\})$ variables can still exist.

For example, with the input $D = \{5, 8, 13, 22, 29\}$ the number of variables is reduced from 78 to 56 via Algorithm 1. On the other hand, the savings is very limited for $D = \{1, 1, 4, 15, 27, 40\}$, with 87 after filtering instead of 89 originally.

**Algorithm 1** Determines which points can be part of a solution. Note that $C$, $T$ and $P$ are sets, with their usual operations.

**function** VALID_POINTS($distances$)
    $C = \{0\}$
    **for** $d \in distances$ **do**
        $T = \{\}$
        **for** $p \in C$ **do**
            $T = T \cup \{p - d, p + d\}$
        $C = C \cup T$
    $P = \{p \,|\, p \in C, p \geq 0\}$
    **return** $P$

### 3.2 Reformulation-Linearization Technique

Available QCQP solvers are usually designed for problems in which all quadratic constraints are convex. This means that the constraint set can be written in the form $x^T Q_i x + a_i x \leq b_i$ for $i \in \{1, \ldots, m\}$, where each matrix $Q_i$ is positive-definite. However, this is not the case because of Equation (4). In this situation, two main families of techniques are usually employed to reformulate the problem: changing the offending matrices to achieve positive definiteness, or linearizing the formulation. We explored both options in preliminary research, and the second option turned out to be the more successful.

The reformulation-linearization technique (RLT) is a family of techniques which is used to produce relaxations of non-convex problems. The resulting relaxation can be tighter at the expense of a larger number of variables and constraints. In the case of a quadratic 0–1 program, it is possible to linearize the problem while maintaining integrality constraints on the variables, hence leading to an ILP. A thorough description of RLT along with several applications in discrete and continuous non-convex optimization can be found in Sherali and Adams (2013), while a more concise presentation can be found in Anstreicher (2009).

A simple application of RLT for a QCQP involves formulating the products of the original $x_i$ variables as new variables $y_{ij}$, and deriving valid linear inequalities on these variables. In the formulation of Equations (3–6), the product $x_i x_j$ appears only if $|i - j| = d$, for some $d \in D'$, such that it is unnecessary to generate all combinations. Each $x_i$ is a binary variable, hence subject to the constraints $x_i \geq 0$ and $x_i \leq 1$. Combining these constraints for $i$ and $j$, and replacing each term $x_i x_j$ by a new variable $y_{ij}$ leads to $\{y_{ij} \geq 0$, $y_{ij} - x_i - x_j \geq -1$, $y_{ij} - x_i \leq 0$, and $y_{ij} - x_j \leq 0\}$. Note that the integrality of the $x_i$ implies the integrality of the $y_{ij}$, such that it is unnecessary to set integrality constraints on all variables. Overall, the following linearized formulation of the MDSP is obtained:

$$\min \quad \sum_{i=0}^{B} x_i \tag{7}$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{i=0}^{B-d} y_{i,i+d} \geq \text{mult}(d) && \forall d \in D' && (8)\\
& y_{ij} - x_i - x_j \geq -1 && 0 \leq i,j \leq B && (9)\\
& y_{ij} - x_i \leq 0 && 0 \leq i,j \leq B && (10)\\
& y_{ij} - x_j \leq 0 && 0 \leq i,j \leq B && (11)\\
& y_{ij} \geq 0 && 0 \leq i,j \leq B && (12)\\
& x_i \in \{0,1\} && 0 \leq i \leq B && (13)
\end{aligned}
$$

The main advantage of this formulation relates to the availability of a wide range of commercial software to solve it, in contrast to the less available quadratic constrained and nonlinear program solvers. On the other hand, some implicit information contained in the original formulation has been lost, which could have been used to strength the model. The performance of this model will be evaluated in Section 5.

# 4 Integer Formulation

The main disadvantage of the quadratic model of Section 3 comes from the use of a binary variable for each possible position in $\{0, \ldots, B\}$. In this section, we propose an alternative ILP formulation which does not rely on a pseudo-polynomial number of binary position variables, but instead uses $k+1$ integer variables $(p_0, \ldots, p_k) \in \mathbb{Z}^{k+1}$, one for the position of each point, as well as distance-to-points assignment variables.

First, in order to reduce symmetry, we impose a predefined order on the positions and make sure that no two points overlap. This can be done via the following constraint:

$$p_i + 1 \leq p_j, \quad 0 \leq i < j \leq k \tag{14}$$

The next step is to introduce variables that can represent distances. Since any distance in $D$ can be assigned between any two points $p_i$ and $p_j$, we introduce a binary variable $x_{ij}^d$ for each $d \in D'$ and each index-pair $(i,j)$ such that $0 \leq i < j \leq k+1$. This binary variable will be set to 1 if and only if the distance $d$ is used between vertices $i$ and $j$. Evidently, no more than one distance should be used between any pair of vertices, leading to the constraint:

$$\sum_{d \in D'} x_{ij}^d \leq 1, \quad 0 \leq i < j \leq k, \tag{15}$$

and each unique distance $d \in D'$ must be used exactly $\text{mult}(d)$ times:

$$\sum_{i=0}^{k-1} \sum_{j=i+1}^{k} x_{ij}^d = \text{mult}(d), \quad \forall d \in D'. \tag{16}$$

Now, if any variable $x_{ij}^d$ is equal to 1, then the distance between $p_i$ and $p_j$ should be equal to $d$. This can be achieved via the constraints

$$p_j \leq p_i + d + B(1 - x_{ij}^d), \quad 0 \leq i < j \leq k, d \in D' \tag{17}$$

$$p_j \geq p_i + dx_{ij}^d, \quad 0 \leq i < j \leq k, d \in D'. \tag{18}$$

As seen in Section 3, $B$ is the sum of all distances in $D$, moreover we can restrict the search to the space of solutions such that the pairwise distance between two points is at most $B$. Equation (17) forces $p_j \leq p_i + d$ when $x_{ij}^d$ equals one, and does not restrain the positions of these points otherwise. Likewise, Equation (18) imposes $p_j \geq p_i + d$ if $x_{ij}^d$ equals one, and does not additionally restrain the positions of $p_i$ and $p_j$ when $x_{ij}^d$ is zero.

Finally, the goal of the problem is to minimize the number of points. To model this objective, we introduce a set of binary variables $z_i \in \{0, 1\}$ for $0 \leq i \leq k$. Each variable $z_i$ is set to 1 if and only if the vertex $p_i$ is used in the solution. Since $p_0$ is fixed at the origin and it is always in the solution, $z_0$ can be fixed as 1. To reduce symmetry, we also impose all active points to be consecutive as follows:

$$z_i \geq z_{i+1}, \quad 0 \leq i < k. \tag{19}$$

Furthermore, if $x_{ij}^d = 1$ for some $d$, then $p_i$ and $p_j$ must be part of the solution and $z_i = z_j = 1$. This can be achieved via a modification of Equation (15):

$$\sum_{d \in D'} x_{ij}^d \leq z_j, \quad 0 \leq i < j \leq k, \tag{20}$$

allowing a distance to be used between $p_i$ and $p_j$ if and only if $z_j = 1$. Moreover $z_j = 1$ implies that $z_i = 1$ for $i < j$ as a consequence of Equation (19).

Finally, the objective is to minimize the number of active vertices – and thus the number of points such that $z_i = 1$. This leads to the following ILP formulation of the MDSP:

$$\min \quad \sum_{i=0}^{k} z_i \tag{21}$$

$$\text{s.t.} \quad \sum_{d \in D'} x_{ij}^d \leq z_j \qquad 0 \leq i < j \leq k \tag{22}$$

$$\sum_{i=0}^{k-1} \sum_{j=i+1}^{k} x_{ij}^d = \text{mult}(d) \qquad d \in D' \tag{23}$$

$$p_j \geq p_i + dx_{ij}^d \qquad 0 \leq i \leq k, d \in D' \tag{24}$$

$$p_j \leq p_i + d + B(1 - x_{ij}^d) \qquad 0 \leq i \leq k, d \in D' \tag{25}$$

$$p_i + 1 \leq p_j \qquad 0 \leq i < j \leq k \tag{26}$$

$$z_i \geq z_{i+1} \qquad 0 \leq i < k \tag{27}$$

$$x_{ij}^d \in \{0, 1\} \qquad 0 \leq i < j \leq k, d \in D' \tag{28}$$

$$p_i \in \mathbb{Z} \qquad 0 \le i \le k \tag{29}$$

$$z_i \in \{0,1\} \qquad 0 \le i \le k. \tag{30}$$

This integer linear programming formulation is clearly more intricate than the quadratic formulation of Equations (3–6). Still, it has the merit to have a polynomial number of constraints and variables.

Note that the size of this ILP model can still grow fairly large, with $1/2(k(k+1))|D'|$ binary variables, $k+1$ integer variables, and $k(k+1)(1+|D'|)+k+|D'|$ constraints. In the worst case, when all distances in $D$ are unique, this models has a cubic number of variables and constraints. For example, if $|D|=|D'|=20$ and all distances are unique, there will be a total of 4200 variables and 8860 constraints, while if $|D|=|D'|=100$, this number grows up to 505000 variables and 1020300 constraints. For this reason, we considered two variations of resolution approach, based on a binary search over the number of active points and two different forms for the subproblems (optimization or feasibility) without the $z_i$ decision variables. These approaches are described in the following.

## 4.1 Binary Search Method

The integer linear formulation presented in the previous section has a polynomial number of variables and constraints, but this number can grow fairly large as the size of the input increases. To deal with this issue, we propose two solution methods, based on a binary search on the number of points in the optimal solution, without need of the $z_i$ variables.

Indeed, the solutions of the MDSP with up to $t+1$ points are all contained in the region described by the following equations:

$$\sum_{i=0}^{t-1} \sum_{j=i+1}^{t} x_{ij}^d = \mathrm{mult}(d) \qquad d \in D' \tag{31}$$

$$\sum_{d \in D'} x_{ij}^d \le 1 \qquad 0 \le i < j \le t \tag{32}$$

$$p_j \ge p_i + d x_{ij}^d \qquad 0 \le i \le t, d \in D' \tag{33}$$

$$p_j \le p_i + d + B(1 - x_{ij}^d) \qquad 0 \le i \le t, d \in D' \tag{34}$$

$$p_i + 1 \le p_j \qquad 0 \le i < j \le t \tag{35}$$

$$x_{ij}^d \in \{0,1\} \qquad 0 \le i < j \le t, d \in D' \tag{36}$$

$$p_i \in \mathbb{Z} \qquad 0 \le i \le t \tag{37}$$

This formulation does not use $z_i$ variables and considers that all points as part of the solution, even points that use no distances whatsoever. Finding a feasible solution of Equations (31–37) means that it is possible to assign all distances in $D$ as distances between $t+1$ points in $\mathbb{Z}$. If this region is empty, then there is no way to assign those distances as the distances between $t+1$ integer points or less.

These observations lead to Algorithm 2, a binary search on the number of points in the solution. Recall that the lower bound for any instance of size $k$ is $\lceil 1/2+\sqrt{1/4+2k}\,\rceil$ and a trivial upper bound of $k+1$ (Equation 1). These bounds provide an initial interval for the binary search. The binary search iteratively solves the model of Equations (31–37) with $t = t_{\text{MID}}$ representing the middle of the current search interval. If a feasible solution is found, then the upper bound is set to $t_{\text{MID}}$, since now it is known that a solution with that many points can exist, but it is not known if a solution with less points exists yet. If no feasible solution is found, then the lower bound is updated to $t_{\text{MID}}+1$, as there cannot exist any solution with less than $t_{\text{MID}}+1$ points. After the loop, the algorithm simply outputs the last solution found. This is sufficient, as the final iteration of the binary search already proved that any solution with one less point cannot exist.

**Algorithm 2** Binary search algorithm.

**function** BS-MDSP($D$)
    $k = \text{length}(D)$
    $t_{\text{LB}} = \left\lceil \frac{1}{2} + \sqrt{\frac{1}{4} + 2k} \right\rceil$
    $t_{\text{UB}} = k+1$
    **while** $t_{\text{LB}} < t_{\text{UB}}$ **do**
        $t_{\text{MID}} = \lfloor (t_{\text{LB}} + t_{\text{UB}})/2 \rfloor$
        Solve the formulation for $D$ with $t = t_{\text{MID}}$
        **if** a feasible MDSP solution was found **then**
            Save the solution
            $t_{\text{UB}} = t_{\text{MID}}$
        **else**
            $t_{\text{LB}} = t_{\text{MID}} + 1$
    Output the latest solution found

The feasibility model of Equations (31–37) can be further transformed into an optimization formulation that relies on distributing as many distances as possible among a fixed number of points, instead of checking if a region is empty or not. Such a model can be described as follows:

$$\max \quad \sum_{d \in D} \sum_{i=0}^{t-1} \sum_{j=i+1}^{t} x_{ij}^{d} \tag{38}$$

$$\text{s.t.} \quad \sum_{i=0}^{t-1} \sum_{j=i+1}^{t} x_{ij}^{d} \leq \text{mult}(d) \qquad d \in D' \tag{39}$$

$$\text{Equations (32–37)} \tag{40}$$

In this formulation, the constraints on the variables $p_i$ remain identical. The key difference is that Equation (16) is replaced by Equation (39), which does not impose to use all distances anymore. The objective of this model is to maximize the number of distances that can be placed among $t+1$ points. This model can also be used within Algorithm 2. Finding a feasible solution for the MDSP is equivalent to finding a solution of Equations (38–40) with a value equal to the size of the input. If the cardinality of the input is $k$ and the value of the optimal solution of Equations (38–40) is strictly smaller than $k$, then we can conclude that it is impossible to distribute that many distances among $t+1$ points. Otherwise, if the optimal solution has a value of $k$, then the variables $p_i$ define a feasible solution of the MDSP, but a smaller one may exist.

Algorithm 2 can work with both models presented in this section. The major difference is that the model of Equations (38–40) is never infeasible, as setting all $x_{ij}^d$ to zero and all $p_i = i$ is always a feasible solution. Therefore, $t_{\text{UB}}$ is set to $t_{\text{MID}}$ only if the optimal solution of the current model has value $k$, and a smaller solution needs to be found or proven to be nonexistent. If the optimal solution value is less than $k$, then $t_{\text{LB}}$ is set to $t_{\text{MID}}+1$. This goes on until a solution is proven to be optimal, *i.e.*, when $t_{\text{LB}} > t_{\text{UB}}$, or when a predefined time limit is attained.

# 5 Computational Experiments

In this section, we discuss all our computational experiments with the models and solution approaches of Sections 3 and 4. All algorithms described in this paper have been implemented in C++. The integer and quadratic programming models have been solved using CPLEX 12.6.3, with default settings. All experiments were done on a server with an Intel Xeon 3.07GHz processor, running Linux.

Our first experiment consists in comparing a direct resolution of the quadratic Model (3–6) with the resolution of the linearized Model (7–13). As mentioned in Section 3, we first tested a convexification of the offending matrices for the quadratic formulation, which did not yield satisfying solutions. Yet, CPLEX also recently made available a feature to perform the convexification of a non-convex QCQP by applying the first approach of Billionnet and Elloumi (2007). As this latter approach was more successful, the tests presented in this article for the quadratic formulation use directly the convexification provided by CPLEX library. Then, we compare these results with those of the compact integer programming formulation of Equations (21–30), as well as Algorithm 2 using Model (31–37), and Algorithm 2 using Model (38–40).

## 5.1 Instances

Four different types of instances were randomly generated to conduct these tests. Most of the instances were generated by creating a set of points in a line (with a maximum distance between consecutive points), calculating all pairwise distances and then making appropriate modifications. Instance sizes were chosen to test the models and algorithms presented in the previous sections and to determine their performance limit, i.e., when they begin to fail solving the problem within a reasonable CPU time budget.

A total of 210 instances were generated to be used among all tests. Each instance was named following the convention *type-constants-id*, where the constants are the numbers used to generate the instance, and the identifier is a sequential number. The instances are defined as follows:

- Full instances: these are PDP instances, that is, instances of size $k = n(n-1)/2$ for some $n \in \mathbb{N}$, with optimal solution $P$ of cardinality $n$. They are complete instances, in the sense that no distances are missing. These instances were generated by sampling $n-1$ integers between 1 and $m$ as the distances between consecutive points on the line, and calculating all pairwise distances. Five of these instances were generated for each pair between $n = 5, 6, 7, 8, 9, 10$ and $m = 15, 30$, for a total of 40 instances. These instances are named following the pattern `full-`$n$`-`$m$`-id`.

- Missing distance instances: these instances contain a number of distances which is halfway between that of a full instance with either $n-1$ or $n$ points. Each instance was generated by randomly removing $(k_2 - k_1)/2$ distances from a Full instance with $n$ points, where $k_1 = n(n-1)/2$ and $k_2 = n(n+1)/2$. The optimal solution value is known, being equal to $n$. Five instances were generated for each pair of $n = 5, 6, 7, 8, 9, 10$ and $m = 15, 30$ (for the underlying full instance), leading to a total of 40 instances. These instances are named according to the pattern `miss-`$(n+1)$`-`$m$`-`$l$`-id`.

- Joint instances: these are the concatenation of two full instances generated from pair of integers $(n_1, m_1)$ and $(n_2, m_2)$. By construction, we can infer an upper bound of $n_1 + n_2 - 1$. Three of these instances were generated for each pair between $(n_1, n_2) = (5, 5), (6, 5), (7, 5), (8, 5), (9, 5), (10, 5)$ and $(m_1, m_2) = (15, 15), (30, 30)$, for a total of 24 instances. These instances are named following the pattern `joint-`$n_1$`-`$m_1$`-`$n_2$`-`$m_2$`-id`.

- Random instances: the instances consist of $k$ integers uniformly sampled in $[1, d]$. The distances $d$ were chosen as approximations to the maximum distances found in the previous distances. The upper bound for these instances is $k$. Three instances were generated for each pair between $k = 5, 7, 10, 15, 20, 25$ and $d = 75, 110, 200$ for a total of 36 instances. These instances are named following the pattern `drand-`$k$`-`$d$`-id`.

The choice of these instance types was driven by the necessity of having a diversified testbed, with upper bounds that differ by construction. We also note that specialized PDP algorithms could solve the "Full" instances efficiently, but not the other cases. As the goal of these experiments is to study solution algorithms for the MDSP, these instances are mainly used as an extreme case, in which the solutions are very "dense": all pairwise distances between points being used.

### 5.2 Comparison of MDSP methods

The MDSP mathematical programming formulations compared in this computational experiment are: the quadratic programming model with filtered variables (3–6), its linearization-reformulation (7–13), the integer programming model (21–30), Algorithm 2 using Model (31–37) and Algorithm 2 using Model (38–40). These methods will be referred to as `QP`, `RLT`, `IP`, `FEAS` and `MAX`, respectively. Each method was given 3600 seconds to run each instance a single time. All methods use the information about the trivial upper bound from Equation (1). This means that for all methods, a trivial feasible solution was provided to the solver.

A summary of the results, aggregated for each group of instances with the same size and characteristics, is presented in Table 1. From left to right, the columns present the trivial average gap, obtained from the trivial lower and upper bounds, as well as the average gap and time of each method. Some runs which did not reach the time limit of 3600 seconds and nevertheless have not obtained the optimal solution raised an "out of memory" exception. When this is the case for a given instance, we considered its running time as the time limit (3600 seconds). Furthermore the detailed results for all instances are presented in Appendix A, in Tables 3-6.

| | | Trivial | QP | | RLT | | IP | | FEAS | | MAX | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Set | Size | Gap | Gap | Time | Gap | Time | Gap | Time | Gap | Time | Gap | Time |
| full | 5 | 54.5% | 5.0% | 3286.0 | 0.0% | 1964.9 | 0.0% | 0.7 | 0.0% | 0.3 | 0.0% | 0.6 |
| | 6 | 62.5% | 37.0% | 3600.0 | 12.5% | 3600.0 | 0.0% | 20.0 | 0.0% | 12.4 | 0.0% | 11.3 |
| | 7 | 68.2% | 61.2% | 3600.0 | 50.7% | 3600.0 | 0.0% | 953.3 | 0.0% | 378.6 | 0.0% | 351.3 |
| | 8 | 72.4% | 72.4% | 3600.0 | 72.4% | 3600.0 | 46.1% | 3600.0 | 28.5% | 3300.3 | 23.2% | 3558.8 |
| | 9 | 75.7% | 75.7% | 3600.0 | 75.7% | 3600.0 | 52.6% | 3600.0 | 39.8% | 3600.0 | 47.0% | 3600.0 |
| | 10 | 78.3% | 78.3% | 3600.0 | 78.3% | 3600.0 | 62.5% | 3600.0 | 48.5% | 3600.0 | 50.1% | 3600.0 |
| miss | 5 | 44.4% | 0.0% | 3102.7 | 0.0% | 1251.1 | 0.0% | 0.2 | 0.0% | 0.2 | 0.0% | 0.2 |
| | 6 | 57.1% | 24.3% | 3600.0 | 30.7% | 3600.0 | 0.0% | 10.4 | 0.0% | 4.6 | 0.0% | 10.9 |
| | 7 | 63.2% | 42.3% | 3600.0 | 43.8% | 3600.0 | 3.0% | 908.9 | 3.0% | 743.0 | 3.0% | 954.1 |
| | 8 | 69.2% | 69.2% | 3600.0 | 69.2% | 3600.0 | 30.2% | 3017.8 | 27.1% | 3404.0 | 27.9% | 3287.8 |
| | 9 | 72.7% | 72.7% | 3600.0 | 72.7% | 3600.0 | 51.7% | 3600.0 | 40.4% | 3600.0 | 37.3% | 3600.0 |
| | 10 | 76.2% | 76.2% | 3600.0 | 76.2% | 3600.0 | 60.9% | 3600.0 | 48.0% | 3600.0 | 51.7% | 3285.2 |
| joint | 5 | 66.7% | 50.3% | 3600.0 | 42.2% | 3600.0 | 18.3% | 2417.7 | 18.3% | 1821.4 | 15.0% | 1987.3 |
| | 6 | 69.2% | 66.5% | 3600.0 | 64.8% | 3600.0 | 31.2% | 3600.0 | 21.1% | 3469.5 | 25.1% | 3107.1 |
| | 7 | 71.9% | 71.9% | 3600.0 | 71.9% | 3600.0 | 38.1% | 3600.0 | 33.1% | 3600.0 | 30.2% | 3600.0 |
| | 8 | 74.4% | 74.4% | 3600.0 | 74.4% | 3600.0 | 47.7% | 3600.0 | 43.5% | 3600.0 | 40.7% | 3600.0 |
| | 9 | 76.6% | 76.6% | 3600.0 | 76.6% | 3600.0 | 66.2% | 3600.0 | 35.8% | 3600.0 | 41.9% | 3600.0 |
| | 10 | 80.4% | 80.4% | 3600.0 | 80.4% | 3600.0 | 72.2% | 3600.0 | 52.4% | 3600.0 | 61.1% | 3600.0 |
| drand | 5 | 33.3% | 0.0% | 195.1 | 0.0% | 2.0 | 0.0% | 0.1 | 0.0% | 0.1 | 0.0% | 0.1 |
| | 7 | 37.5% | 29.2% | 3600.0 | 27.9% | 3600.0 | 0.0% | 33.9 | 0.0% | 32.4 | 0.0% | 81.8 |
| | 10 | 54.5% | 47.7% | 3600.0 | 47.0% | 3600.0 | 2.5% | 985.3 | 6.1% | 612.0 | 6.1% | 1182.6 |
| | 15 | 62.5% | 61.3% | 3600.0 | 61.3% | 3600.0 | 36.0% | 3600.0 | 31.4% | 3138.6 | 37.2% | 3405.4 |
| | 20 | 66.7% | 66.7% | 3600.0 | 66.7% | 3600.0 | 55.1% | 3600.0 | 50.0% | 3600.0 | 45.6% | 3600.0 |
| | 25 | 69.2% | 69.2% | 3600.0 | 69.2% | 3600.0 | 56.7% | 3600.0 | 52.9% | 3600.0 | 50.8% | 3600.0 |
| Average | | 64.3% | 53.0% | 3415.4 | 51.1% | 3256.1 | 28.9% | 2180.9 | 23.1% | 2092.5 | 23.5% | 2125.7 |
| Solved | | | 32 | | 45 | | 90 | | 95 | | 97 | |

Table 1: Performance of the five MDSP solution methods

In these experiments, the quadratic programming model with filtered variables (3–6) and its linearization-reformulation (7–13) were outperformed by the other methods. Out of the 210 instances, `QP` could solve only 32 instances, while this number rose up to 45 for `RLT`. `QP` was not able to improve the trivial bounds for 140 instances, reaching an average gap of 53.0%. Similarly, `RLT` did not improve the trivial bounds for 141 instances, reaching an average gap of 51.1%.

These are only small improvements over the average trivial gap of 64.3%. Finally, RLT resulted in an "out of memory" status for 10 instances, and QP did not reach this status.

Solving the proposed IP model, instead, allows to find optimal solutions for 90 out of the 210 instances. This is a clear improvement over the QP and RLT approaches. With an average gap of 28.9%, IP improved the trivial bounds on the majority of instances (all except 8). Moreover, the use of a binary search as in Algorithm 2, using FEAS and MAX, allows to solve to optimality a few additional instances. These methods were able to obtain the optimal solution for 95 and 97 instances, respectively. FEAS obtained an average gap of 23.1% and was not able to improve over the trivial bounds on only 2 instances. MAX obtained an average gap of 23.5% and was not able to improve over the trivial bounds only on 6 instances. For all three approaches, no instance resulted in an "out of memory" status.

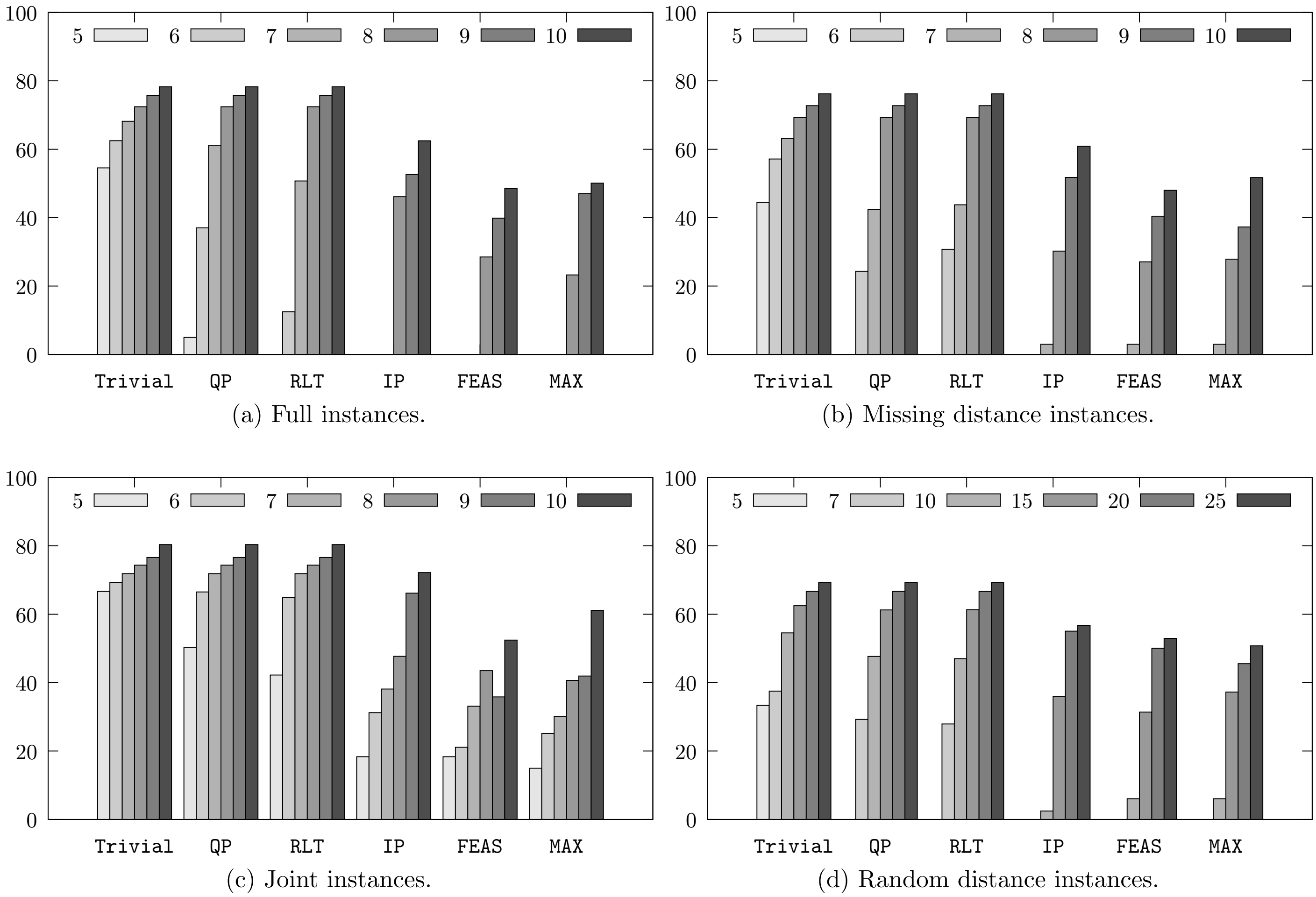
As highlighted by these experiments, the IP, FEAS and MAX approaches, based on the compact formulation of Section 4, outperform the QP and RLT approaches. Moreover, the FEAS and MAX approaches appear to provide better final gaps and solved to optimality a few more instances. To confirm these observations, we conducted a Friedman test, considering the final gap of each method on the 210 instances. This test confirms with high confidence ($p < 10^{-16}$) that significant differences of performance exist between the five proposed methods.

Then, we used the post-hoc Wilcoxon-Nemenyi-McDonald-Thompson test (Hollander et al., 2013) to perform pairwise comparisons between the methods. The p-values resulting from this analysis are presented in Table 2. These statistical analyses confirm our previous observations: the final gap values obtained by IP, FEAS and MAX are all significantly better than those of QP and RLT ($p < 0.00001$ in all cases). FEAS and MAX perform better than IP ($p \approx 0.02$), and no significant difference can be highlighted, at least for the current instances, between FEAS and MAX. In this sense, the choice of the model variant (feasibility or optimization) in the binary search algorithm of Section 4.1 has a more minor impact.

| | QP | RLT | IP | FEAS | MAX |
|---|---|---|---|---|---|
| QP | — | 0.73684 | $< 0.00001$ | $< 0.00001$ | $< 0.00001$ |
| RLT | | — | $< 0.00001$ | $< 0.00001$ | $< 0.00001$ |
| IP | | | — | 0.00197 | 0.01919 |
| FEAS | | | | — | 0.96517 |
| MAX | | | | | — |

Table 2: P-values obtained from the post-hoc analysis

Figure 2 completes this analysis with a graphical comparison of the average percentage gap obtained by all approaches. Finally, Figure 3 compares the most frequently obtained solution status for each method: instances solved to optimality are represented by a light gray color, instances for which the approach was able to improve over the trivial bound are represented by a medium gray color and instances which there was no improvement are represented by a dark gray color. As visible in both figures again, FEAS and MAX outperform the other approaches.



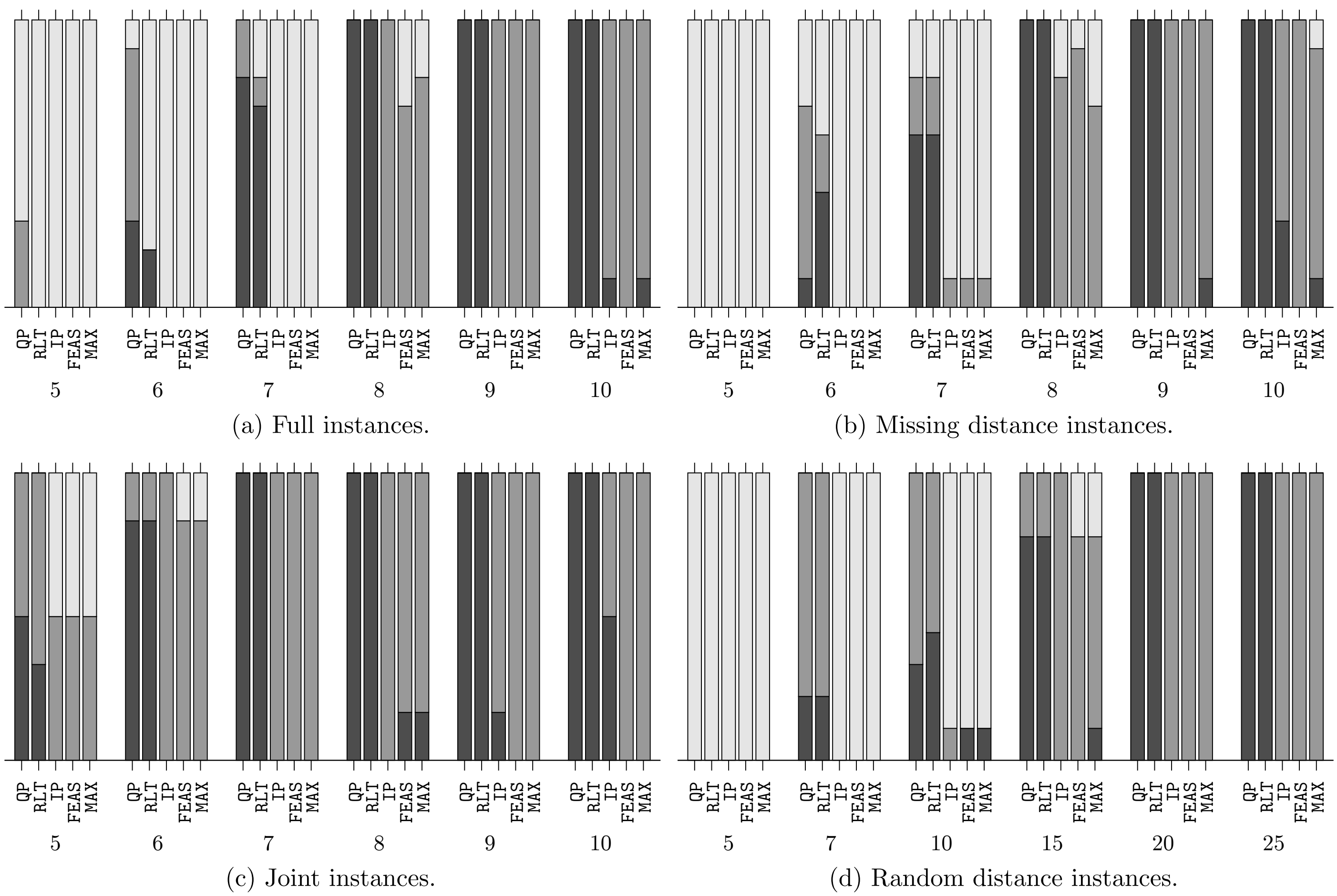

(a) Full instances.

(b) Missing distance instances.

(c) Joint instances.

(d) Random distance instances.

Figure 2: Comparison of the average percent optimality gap, for each method and instance set. The larger the gap, the worse is the solution.



(a) Full instances.

(b) Missing distance instances.

(c) Joint instances.

(d) Random distance instances.

Figure 3: Solution statuses for each method and instance set. Instances solved to optimality are represented by a light gray color, the ones for which there was an improvement over the trivial bounds are represented by a medium gray color, and the ones which there was no improvement are represented by a dark gray color.

Finally, we observed that the proposed algorithms sometimes find different optimal solutions. For example, for an instance with $D = \{16, 31, 40, 57, 57, 61, 65, 69, 69, 75\}$, the `IP` method finds the solution $P_{\texttt{IP}} = \{0, 4, 12, 16, 69, 73, 113, 144\}$, the `FEAS` method finds the solution $P_{\texttt{FEAS}} = \{0, 6, 14, 18, 31, 71, 75, 87\}$ and the `MAX` method finds the solution $P_{\texttt{MAX}} = \{0, 2, 40, 59, 63, 71, 75, 128\}$.

## 6 Conclusions and Research Perspectives

In this work, we have proposed new mathematical programming formulations for the minimum distance superset problem: a quadratically constrained $0-1$ formulation containing a pseudo-polynomial number of variables (Section 3), and a compact integer formulation (Section 4). For the first formulation, we investigated the use of a reformulation-linearization technique, leading to an integer linear program which is solvable by a regular solver. For the second compact formulation, two solution approaches involving binary searches on the number of points in the solution were introduced. These approaches proved to be successful, as they generate significantly better results in comparison to the other algorithms.

For future works, the research perspectives are numerous, we recommend to:

- Study the approximability of the MDSP. Such a result may exist, as Cieliebak has proven it for a similar problem (Cieliebak et al., 2003).
- Study the symmetry in MDSP solutions. Symmetry breaking constraints may improve the efficiency of all models considered in this work, by significantly reducing the search space.
- Improve Algorithm 1 or find a better substitute. Reducing the number of variables in the quadratic $0-1$ model will drastically improve its efficiency, especially if it can be reduced to a polynomial size on the input.
- Develop heuristics which can improve the trivial lower bounds for a given MDSP instance. This should improve the efficiency of all presented models, especially the binary search based methods.
- Keep on progressing towards more advanced mathematic programming techniques for the resolution of Model (21)-(30).
- Investigate further the application of combinatorial algorithms, inspired by Skiena et al. (1990), for the MDSP.

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# A Detailed Results

In Tables 3-6, we present the trivial lower and upper bounds and, for each method, the lower and upper bounds obtained, the remaining gap and its running time in seconds. We also indicate with a "t" the entries which did not improve over the trivial lower bound. Furthermore, entries which did not reach the time limit of 3600 seconds and nevertheless have not obtained the optimal solution raised an out of memory exception. These instances are indicated with an "—".

Table 3: Results for full instances

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| full-5-15-1 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 156.8 | 5 | 5 | 0.0% | 1.0 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.2 |
| full-5-15-2 | 5 | 11 | 54.5% | 5 | 6 | 16.7% | 3600.0 | 5 | 5 | 0.0% | 2720.9 | 5 | 5 | 0.0% | 0.5 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.3 |
| full-5-15-3 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 0.6 | 5 | 5 | 0.0% | 0.4 | 5 | 5 | 0.0% | 0.5 |
| full-5-15-4 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 217.1 | 5 | 5 | 0.0% | 0.4 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.3 |
| full-5-15-5 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 459.6 | 5 | 5 | 0.0% | 5.0 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.2 |
| full-5-30-1 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 1.9 | 5 | 5 | 0.0% | 0.6 | 5 | 5 | 0.0% | 2.0 |
| full-5-30-2 | 5 | 11 | 54.5% | 5 | 6 | 16.7% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 0.8 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 1.2 |
| full-5-30-3 | 5 | 11 | 54.5% | 5 | 6 | 16.7% | 3600.0 | 5 | 5 | 0.0% | 2072.0 | 5 | 5 | 0.0% | 0.5 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.4 |
| full-5-30-4 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 76.9 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.6 |
| full-5-30-5 | 5 | 11 | 54.5% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 0.7 | 5 | 5 | 0.0% | 0.4 | 5 | 5 | 0.0% | 0.5 |
| full-6-15-1 | 6 | 16 | 62.5% | 6 | 7 | 14.3% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 6.6 | 6 | 6 | 0.0% | 1.6 | 6 | 6 | 0.0% | 11.1 |
| full-6-15-2 | 6 | 16 | 62.5% | 6 | 12 | 50.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 3.5 | 6 | 6 | 0.0% | 10.7 | 6 | 6 | 0.0% | 7.8 |
| full-6-15-3 | 6 | 16 | 62.5% | 6 | 8 | 25.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 5.0 | 6 | 6 | 0.0% | 3.4 | 6 | 6 | 0.0% | 4.1 |
| full-6-15-4 | 6 | 16 | 62.5% | 6 | 13 | 53.8% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 5.9 | 6 | 6 | 0.0% | 4.8 | 6 | 6 | 0.0% | 7.9 |
| full-6-15-5 | 6 | 16 | 62.5% | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 5.8 | 6 | 6 | 0.0% | 3.1 | 6 | 6 | 0.0% | 3.7 |
| full-6-30-1 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 77.2 | 6 | 6 | 0.0% | 45.6 | 6 | 6 | 0.0% | 11.5 |
| full-6-30-2 | 6 | 16 | 62.5% | 6 | 8 | 25.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 8.5 | 6 | 6 | 0.0% | 4.7 | 6 | 6 | 0.0% | 4.8 |
| full-6-30-3 | 6 | 16 | 62.5% | 6 | 7 | 14.3% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 5.3 | 6 | 6 | 0.0% | 2.5 | 6 | 6 | 0.0% | 1.6 |
| full-6-30-4 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 6 | 6 | 0.0% | 59.0 | 6 | 6 | 0.0% | 19.3 | 6 | 6 | 0.0% | 29.0 |
| full-6-30-5 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 6 | 6 | 0.0% | 23.0 | 6 | 6 | 0.0% | 28.0 | 6 | 6 | 0.0% | 31.4 |
| full-7-15-1 | 7 | 22 | 68.2% | 7 | 11 | 36.4% | 3600.0 | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 58.0 | 7 | 7 | 0.0% | 5.6 | 7 | 7 | 0.0% | 8.2 |
| full-7-15-2 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 3600.0 | 7 | 7 | 0.0% | 384.6 | 7 | 7 | 0.0% | 22.2 | 7 | 7 | 0.0% | 295.7 |
| full-7-15-3 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 3600.0 | 7 | 7 | 0.0% | 192.5 | 7 | 7 | 0.0% | 185.5 | 7 | 7 | 0.0% | 154.8 |
| full-7-15-4 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 1182.0 | 7 | 7 | 0.0% | 875.0 | 7 | 7 | 0.0% | 492.3 |
| full-7-15-5 | 7 | 22 | 68.2% | 7 | 10 | 30.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 | 7 | 7 | 0.0% | 202.2 | 7 | 7 | 0.0% | 101.7 | 7 | 7 | 0.0% | 38.2 |
| full-7-30-1 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 658.8 | 7 | 7 | 0.0% | 1854.7 | 7 | 7 | 0.0% | 694.9 |
| full-7-30-2 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 2391.7 | 7 | 7 | 0.0% | 64.9 | 7 | 7 | 0.0% | 908.0 |

*Continue on next page*

Table 3: Results for full instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| full-7-30-3 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 1157.3 | 7 | 7 | 0.0% | 613.4 | 7 | 7 | 0.0% | 15.1 |
| full-7-30-4 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 2841.2 | 7 | 7 | 0.0% | 34.0 | 7 | 7 | 0.0% | 136.3 |
| full-7-30-5 | 7 | 22 | 68.2% | 7 | 22 | 68.2% | 3600.0 t | 7 | 22 | 68.2% | 3600.0 t | 7 | 7 | 0.0% | 464.7 | 7 | 7 | 0.0% | 29.1 | 7 | 7 | 0.0% | 769.6 |
| full-8-15-1 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 14 | 42.9% | 3600.0 | 8 | 8 | 0.0% | 3082.4 | 8 | 10 | 20.0% | 3600.0 |
| full-8-15-2 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 13 | 38.5% | 3600.0 | 8 | 13 | 38.5% | 3600.0 | 8 | 10 | 20.0% | 3600.0 |
| full-8-15-3 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 11 | 27.3% | 3600.0 | 8 | 8 | 0.0% | 1815.8 | 8 | 8 | 0.0% | 3554.3 |
| full-8-15-4 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 12 | 33.3% | 3600.0 | 8 | 8 | 0.0% | 2904.9 | 8 | 10 | 20.0% | 3600.0 |
| full-8-15-5 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 12 | 33.3% | 3600.0 | 8 | 10 | 20.0% | 3600.0 | 8 | 13 | 38.5% | 3600.0 |
| full-8-30-1 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 14 | 42.9% | 3600.0 | 8 | 13 | 38.5% | 3600.0 | 8 | 8 | 0.0% | 3233.4 |
| full-8-30-2 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 25 | 68.0% | 3600.0 | 8 | 18 | 55.6% | 3600.0 | 8 | 10 | 20.0% | 3600.0 |
| full-8-30-3 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 16 | 50.0% | 3600.0 | 8 | 13 | 38.5% | 3600.0 | 8 | 10 | 20.0% | 3600.0 |
| full-8-30-4 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 23 | 65.2% | 3600.0 | 8 | 13 | 38.5% | 3600.0 | 8 | 13 | 38.5% | 3600.0 |
| full-8-30-5 | 8 | 29 | 72.4% | 8 | 29 | 72.4% | 3600.0 t | 8 | 29 | 72.4% | 3600.0 t | 8 | 20 | 60.0% | 3600.0 | 8 | 18 | 55.6% | 3600.0 | 8 | 18 | 55.6% | 3600.0 |
| full-9-15-1 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 14 | 35.7% | 3600.0 | 9 | 16 | 43.8% | 3600.0 | 9 | 16 | 43.8% | 3600.0 |
| full-9-15-2 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 15 | 40.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 | 9 | 16 | 43.8% | 3600.0 |
| full-9-15-3 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 23 | 60.9% | 3600.0 | 9 | 16 | 43.8% | 3600.0 | 9 | 16 | 43.8% | 3600.0 |
| full-9-15-4 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 20 | 55.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 | 9 | 16 | 43.8% | 3600.0 |
| full-9-15-5 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 13 | 30.8% | 3600.0 | 9 | 12 | 25.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 |
| full-9-30-1 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 22 | 59.1% | 3600.0 | 9 | 16 | 43.8% | 3600.0 | 9 | 23 | 60.9% | 3600.0 |
| full-9-30-2 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 18 | 50.0% | 3600.0 | 9 | 16 | 43.8% | 3600.0 | 9 | 16 | 43.8% | 3600.0 |
| full-9-30-3 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 36 | 75.0% | 3600.0 | 9 | 16 | 43.8% | 3600.0 | 9 | 23 | 60.9% | 3600.0 |
| full-9-30-4 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 24 | 62.5% | 3600.0 | 9 | 16 | 43.8% | 3600.0 | 9 | 23 | 60.9% | 3600.0 |
| full-9-30-5 | 9 | 37 | 75.7% | 9 | 37 | 75.7% | 3600.0 t | 9 | 37 | 75.7% | 3600.0 t | 9 | 21 | 57.1% | 3600.0 | 9 | 23 | 60.9% | 3600.0 | 9 | 16 | 43.8% | 3600.0 |
| full-10-15-1 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 | 10 | 19 | 47.4% | 3600.0 | 10 | 28 | 64.3% | 3600.0 |
| full-10-15-2 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 17 | 41.2% | 3600.0 | 10 | 14 | 28.6% | 3600.0 | 10 | 14 | 28.6% | 3600.0 |
| full-10-15-3 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 20 | 50.0% | 3600.0 | 10 | 14 | 28.6% | 3600.0 | 10 | 19 | 47.4% | 3600.0 |
| full-10-15-4 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 27 | 63.0% | 3600.0 | 10 | 14 | 28.6% | 3600.0 | 10 | 19 | 47.4% | 3600.0 |

*Continue on next page*

Table 3: Results for full instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| full-10-15-5 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 21 | 52.4% | 3600.0 | 10 | 19 | 47.4% | 3600.0 | 10 | 14 | 28.6% | 3600.0 |
| full-10-30-1 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 26 | 61.5% | 3600.0 | 10 | 28 | 64.3% | 3600.0 | 10 | 19 | 47.4% | 3600.0 |
| full-10-30-2 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 26 | 61.5% | 3600.0 | 10 | 28 | 64.3% | 3600.0 | 10 | 19 | 47.4% | 3600.0 |
| full-10-30-3 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | 3600.0 t | 10 | 27 | 63.0% | 3600.0 | 10 | 19 | 47.4% | 3600.0 | 10 | 28 | 64.3% | 3600.0 |
| full-10-30-4 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | — t | 10 | 46 | 78.3% | 3600.0 t | 10 | 28 | 64.3% | 3600.0 | 10 | 46 | 78.3% | 3600.0 t |
| full-10-30-5 | 10 | 46 | 78.3% | 10 | 46 | 78.3% | 3600.0 t | 10 | 46 | 78.3% | — t | 10 | 45 | 77.8% | 3600.0 | 10 | 28 | 64.3% | 3600.0 | 10 | 19 | 47.4% | 3600.0 |
| Average | 7.5 | 26.8 | 68.6% | 7.5 | 24.7 | 54.9% | 3547.7 | 7.5 | 23.8 | 48.3% | 3318.1 | 7.5 | 14.1 | 26.9% | 1962.3 | 7.5 | 11.1 | 19.5% | 1815.3 | 7.5 | 11.5 | 20.1% | 1853.7 |
| Solved | | | | | | 8 | | | | 20 | | | | 30 | | | | 33 | | | | 32 | |
| Trivial | | | | | | 41 | | | | 39 | | | | 1 | | | | 0 | | | | 1 | |
| Out of Memory | | | | | | 0 | | | | 2 | | | | 0 | | | | 0 | | | | 0 | |

Table 4: Results for missing distance instances

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| miss-5-15-2-1 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 375.3 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.2 |
| miss-5-15-2-2 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 6.2 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.1 |
| miss-5-15-2-3 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 238.9 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.1 |
| miss-5-15-2-4 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3447.0 | 5 | 5 | 0.0% | 39.3 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.2 |
| miss-5-15-2-5 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 2373.9 | 5 | 5 | 0.0% | 20.0 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.2 |
| miss-5-30-2-1 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 19.0 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.2 |
| miss-5-30-2-2 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 1018.2 | 5 | 5 | 0.0% | 0.4 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.5 |
| miss-5-30-2-3 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 0.4 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.4 |
| miss-5-30-2-4 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.4 |
| miss-5-30-2-5 | 5 | 9 | 44.4% | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 3600.0 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.2 | 5 | 5 | 0.0% | 0.2 |
| miss-6-15-2-1 | 6 | 14 | 57.1% | 6 | 13 | 53.8% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 6 | 0.0% | 23.9 | 6 | 6 | 0.0% | 7.6 | 6 | 6 | 0.0% | 14.1 |
| miss-6-15-2-2 | 6 | 14 | 57.1% | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 1.1 | 6 | 6 | 0.0% | 1.4 | 6 | 6 | 0.0% | 1.9 |

*Continue on next page*

Table 4: Results for missing distance instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| miss-6-15-2-3 | 6 | 14 | 57.1% | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 1.7 | 6 | 6 | 0.0% | 3.4 | 6 | 6 | 0.0% | 6.7 |
| miss-6-15-2-4 | 6 | 14 | 57.1% | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 9.1 | 6 | 6 | 0.0% | 1.7 | 6 | 6 | 0.0% | 2.9 |
| miss-6-15-2-5 | 6 | 14 | 57.1% | 6 | 7 | 14.3% | 3600.0 | 6 | 6 | 0.0% | 3600.0 | 6 | 6 | 0.0% | 2.0 | 6 | 6 | 0.0% | 1.7 | 6 | 6 | 0.0% | 10.7 |
| miss-6-30-2-1 | 6 | 14 | 57.1% | 6 | 8 | 25.0% | 3600.0 | 6 | 14 | 57.1% | 3600.0 t | 6 | 6 | 0.0% | 5.1 | 6 | 6 | 0.0% | 4.3 | 6 | 6 | 0.0% | 39.1 |
| miss-6-30-2-2 | 6 | 14 | 57.1% | 6 | 7 | 14.3% | 3600.0 | 6 | 9 | 33.3% | 3600.0 | 6 | 6 | 0.0% | 22.6 | 6 | 6 | 0.0% | 14.7 | 6 | 6 | 0.0% | 7.4 |
| miss-6-30-2-3 | 6 | 14 | 57.1% | 6 | 11 | 45.5% | 3600.0 | 6 | 14 | 57.1% | 3600.0 t | 6 | 6 | 0.0% | 8.8 | 6 | 6 | 0.0% | 2.0 | 6 | 6 | 0.0% | 2.5 |
| miss-6-30-2-4 | 6 | 14 | 57.1% | 6 | 14 | 57.1% | 3600.0 t | 6 | 14 | 57.1% | 3600.0 t | 6 | 6 | 0.0% | 0.6 | 6 | 6 | 0.0% | 1.5 | 6 | 6 | 0.0% | 3.9 |
| miss-6-30-2-5 | 6 | 14 | 57.1% | 6 | 9 | 33.3% | 3600.0 | 6 | 14 | 57.1% | 3600.0 t | 6 | 6 | 0.0% | 29.1 | 6 | 6 | 0.0% | 7.5 | 6 | 6 | 0.0% | 19.5 |
| miss-7-15-3-1 | 7 | 19 | 63.2% | 7 | 9 | 22.2% | 3600.0 | 7 | 7 | 0.0% | 3600.0 | 7 | 7 | 0.0% | 55.3 | 7 | 7 | 0.0% | 13.5 | 7 | 7 | 0.0% | 47.1 |
| miss-7-15-3-2 | 7 | 19 | 63.2% | 7 | 7 | 0.0% | 3600.0 | 7 | 7 | 0.0% | 3600.0 | 7 | 7 | 0.0% | 297.1 | 7 | 7 | 0.0% | 14.1 | 7 | 7 | 0.0% | 119.2 |
| miss-7-15-3-3 | 7 | 19 | 63.2% | 7 | 9 | 22.2% | 3600.0 | 7 | 11 | 36.4% | 3600.0 | 7 | 7 | 0.0% | 214.4 | 7 | 7 | 0.0% | 35.8 | 7 | 7 | 0.0% | 869.3 |
| miss-7-15-3-4 | 7 | 19 | 63.2% | 7 | 7 | 0.0% | 3600.0 | 7 | 9 | 22.2% | 3600.0 | 7 | 7 | 0.0% | 188.5 | 7 | 7 | 0.0% | 82.1 | 7 | 7 | 0.0% | 30.4 |
| miss-7-15-3-5 | 7 | 19 | 63.2% | 7 | 19 | 63.2% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 t | 7 | 7 | 0.0% | 71.8 | 7 | 7 | 0.0% | 100.5 | 7 | 7 | 0.0% | 218.5 |
| miss-7-30-3-1 | 7 | 19 | 63.2% | 7 | 19 | 63.2% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 t | 7 | 7 | 0.0% | 721.8 | 7 | 7 | 0.0% | 242.3 | 7 | 7 | 0.0% | 706.0 |
| miss-7-30-3-2 | 7 | 19 | 63.2% | 7 | 19 | 63.2% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 t | 7 | 7 | 0.0% | 3451.0 | 7 | 10 | 30.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 |
| miss-7-30-3-3 | 7 | 19 | 63.2% | 7 | 19 | 63.2% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 t | 7 | 7 | 0.0% | 244.4 | 7 | 7 | 0.0% | 1455.3 | 7 | 7 | 0.0% | 1183.2 |
| miss-7-30-3-4 | 7 | 19 | 63.2% | 7 | 19 | 63.2% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 t | 7 | 10 | 30.0% | 3600.0 | 7 | 7 | 0.0% | 1206.7 | 7 | 7 | 0.0% | 167.4 |
| miss-7-30-3-5 | 7 | 19 | 63.2% | 7 | 19 | 63.2% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 t | 7 | 7 | 0.0% | 245.0 | 7 | 7 | 0.0% | 679.6 | 7 | 7 | 0.0% | 2600.3 |
| miss-8-15-3-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 13 | 38.5% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 8 | 0.0% | 2011.0 |
| miss-8-15-3-2 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 14 | 42.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| miss-8-15-3-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 8 | 0.0% | 869.6 | 8 | 10 | 20.0% | 3600.0 | 8 | 10 | 20.0% | 3600.0 |
| miss-8-15-3-4 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 10 | 20.0% | 3600.0 | 8 | 9 | 11.1% | 3600.0 | 8 | 8 | 0.0% | 3541.2 |
| miss-8-15-3-5 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 11 | 27.3% | 3600.0 | 8 | 8 | 0.0% | 1640.0 | 8 | 8 | 0.0% | 2125.5 |
| miss-8-30-3-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 13 | 38.5% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 12 | 33.3% | 3600.0 |
| miss-8-30-3-2 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 16 | 50.0% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| miss-8-30-3-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 13 | 38.5% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| miss-8-30-3-4 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 8 | 0.0% | 508.7 | 8 | 10 | 20.0% | 3600.0 | 8 | 12 | 33.3% | 3600.0 |

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Table 4: Results for missing distance instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| miss-8-30-3-5 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 15 | 46.7% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 12 | 33.3% | 3600.0 |
| miss-9-15-4-1 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 11 | 18.2% | 3600.0 | 9 | 15 | 40.0% | 3600.0 | 9 | 15 | 40.0% | 3600.0 |
| miss-9-15-4-2 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 20 | 55.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 |
| miss-9-15-4-3 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 17 | 47.1% | 3600.0 | 9 | 15 | 40.0% | 3600.0 | 9 | 15 | 40.0% | 3600.0 |
| miss-9-15-4-4 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 17 | 47.1% | 3600.0 | 9 | 15 | 40.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 |
| miss-9-15-4-5 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 12 | 25.0% | 3600.0 | 9 | 12 | 25.0% | 3600.0 | 9 | 10 | 10.0% | 3600.0 |
| miss-9-30-4-1 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 26 | 65.4% | 3600.0 | 9 | 21 | 57.1% | 3600.0 | 9 | 15 | 40.0% | 3600.0 |
| miss-9-30-4-2 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 25 | 64.0% | 3600.0 | 9 | 15 | 40.0% | 3600.0 | 9 | 15 | 40.0% | 3600.0 |
| miss-9-30-4-3 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 24 | 62.5% | 3600.0 | 9 | 21 | 57.1% | 3600.0 | 9 | 15 | 40.0% | 3600.0 |
| miss-9-30-4-4 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 28 | 67.9% | 3600.0 | 9 | 15 | 40.0% | 3600.0 | 9 | 15 | 40.0% | 3600.0 |
| miss-9-30-4-5 | 9 | 33 | 72.7% | 9 | 33 | 72.7% | 3600.0 t | 9 | 33 | 72.7% | 3600.0 t | 9 | 26 | 65.4% | 3600.0 | 9 | 15 | 40.0% | 3600.0 | 9 | 33 | 72.7% | 3600.0 t |
| miss-10-15-4-1 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 28 | 64.3% | 3600.0 | 10 | 18 | 44.4% | 3600.0 | 10 | 26 | 61.5% | 3600.0 |
| miss-10-15-4-2 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 23 | 56.5% | 3600.0 | 10 | 18 | 44.4% | 3600.0 | 10 | 26 | 61.5% | 3600.0 |
| miss-10-15-4-3 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 17 | 41.2% | 3600.0 | 10 | 18 | 44.4% | 3600.0 | 10 | 26 | 61.5% | 3600.0 |
| miss-10-15-4-4 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 17 | 41.2% | 3600.0 | 10 | 14 | 28.6% | 3600.0 | 10 | 18 | 44.4% | 3600.0 |
| miss-10-15-4-5 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 15 | 33.3% | 3600.0 | 10 | 18 | 44.4% | 3600.0 | 10 | 10 | 0.0% | 452.2 |
| miss-10-30-4-1 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 32 | 68.8% | 3600.0 | 10 | 26 | 61.5% | 3600.0 | 10 | 26 | 61.5% | 3600.0 |
| miss-10-30-4-2 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | 3600.0 t | 10 | 18 | 44.4% | 3600.0 | 10 | 18 | 44.4% | 3600.0 |
| miss-10-30-4-3 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | — t | 10 | 42 | 76.2% | 3600.0 t | 10 | 26 | 61.5% | 3600.0 | 10 | 18 | 44.4% | 3600.0 |
| miss-10-30-4-4 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | — t | 10 | 40 | 75.0% | 3600.0 | 10 | 26 | 61.5% | 3600.0 | 10 | 42 | 76.2% | 3600.0 t |
| miss-10-30-4-5 | 10 | 42 | 76.2% | 10 | 42 | 76.2% | 3600.0 t | 10 | 42 | 76.2% | — t | 10 | 42 | 76.2% | 3600.0 t | 10 | 18 | 44.4% | 3600.0 | 10 | 26 | 61.5% | 3600.0 |
| Average | 7.5 | 23.8 | 63.8% | 7.5 | 21.6 | 47.5% | 3517.1 | 7.5 | 21.8 | 48.8% | 3187.9 | 7.5 | 13.5 | 24.3% | 1856.2 | 7.5 | 10.9 | 19.7% | 1892.0 | 7.5 | 11.6 | 20.0% | 1856.4 |
| Solved | | | | | | 15 | | | | 16 | | | | 31 | | | | 30 | | | | 33 | |
| Trivial | | | | | | 37 | | | | 40 | | | | 3 | | | | 0 | | | | 2 | |
| Out of Memory | | | | | | 0 | | | | 3 | | | | 0 | | | | 0 | | | | 0 | |

Table 5: Results for joint instances

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| joint-5-15-5-15-1 | 7 | 21 | 66.7% | 7 | 10 | 30.0% | 3600.0 | 7 | 9 | 22.2% | 3600.0 | 8 | 8 | 0.0% | 1189.4 | 8 | 8 | 0.0% | 33.3 | 8 | 8 | 0.0% | 442.6 |
| joint-5-15-5-15-2 | 7 | 21 | 66.7% | 7 | 12 | 41.7% | 3600.0 | 7 | 9 | 22.2% | 3600.0 | 8 | 8 | 0.0% | 2362.2 | 8 | 8 | 0.0% | 64.1 | 8 | 8 | 0.0% | 342.2 |
| joint-5-15-5-15-3 | 7 | 21 | 66.7% | 7 | 10 | 30.0% | 3600.0 | 7 | 8 | 12.5% | 3600.0 | 8 | 8 | 0.0% | 154.6 | 8 | 8 | 0.0% | 31.1 | 8 | 8 | 0.0% | 339.1 |
| joint-5-30-5-30-1 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 13 | 46.2% | 3600.0 | 7 | 10 | 30.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 |
| joint-5-30-5-30-2 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 9 | 22.2% | 3600.0 | 7 | 10 | 30.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 |
| joint-5-30-5-30-3 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 19 | 63.2% | 3600.0 | 7 | 12 | 41.7% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 |
| joint-6-15-5-15-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 10 | 20.0% | 3600.0 | 9 | 9 | 0.0% | 2817.1 | 8 | 12 | 33.3% | 3600.0 |
| joint-6-15-5-15-2 | 8 | 26 | 69.2% | 8 | 17 | 52.9% | 3600.0 | 8 | 14 | 42.9% | 3600.0 | 8 | 9 | 11.1% | 3600.0 | 8 | 10 | 20.0% | 3600.0 | 9 | 9 | 0.0% | 642.4 |
| joint-6-15-5-15-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 12 | 33.3% | 3600.0 | 8 | 10 | 20.0% | 3600.0 | 8 | 9 | 11.1% | 3600.0 |
| joint-6-30-5-30-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 15 | 46.7% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| joint-6-30-5-30-2 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 14 | 42.9% | 3600.0 | 8 | 12 | 33.3% | 3600.0 | 8 | 10 | 20.0% | 3600.0 |
| joint-6-30-5-30-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 12 | 33.3% | 3600.0 | 8 | 10 | 20.0% | 3600.0 | 8 | 12 | 33.3% | 3600.0 |
| joint-7-15-5-15-1 | 9 | 32 | 71.9% | 9 | 32 | 71.9% | 3600.0 t | 9 | 32 | 71.9% | 3600.0 t | 9 | 13 | 30.8% | 3600.0 | 9 | 14 | 35.7% | 3600.0 | 9 | 11 | 18.2% | 3600.0 |
| joint-7-15-5-15-2 | 9 | 32 | 71.9% | 9 | 32 | 71.9% | 3600.0 t | 9 | 32 | 71.9% | 3600.0 t | 9 | 11 | 18.2% | 3600.0 | 9 | 11 | 18.2% | 3600.0 | 9 | 14 | 35.7% | 3600.0 |
| joint-7-15-5-15-3 | 9 | 32 | 71.9% | 9 | 32 | 71.9% | 3600.0 t | 9 | 32 | 71.9% | 3600.0 t | 9 | 12 | 25.0% | 3600.0 | 9 | 11 | 18.2% | 3600.0 | 9 | 11 | 18.2% | 3600.0 |
| joint-7-30-5-30-1 | 9 | 32 | 71.9% | 9 | 32 | 71.9% | 3600.0 t | 9 | 32 | 71.9% | 3600.0 t | 9 | 16 | 43.8% | 3600.0 | 9 | 14 | 35.7% | 3600.0 | 9 | 11 | 18.2% | 3600.0 |
| joint-7-30-5-30-2 | 9 | 32 | 71.9% | 9 | 32 | 71.9% | 3600.0 t | 9 | 32 | 71.9% | 3600.0 t | 9 | 17 | 47.1% | 3600.0 | 9 | 14 | 35.7% | 3600.0 | 9 | 20 | 55.0% | 3600.0 |
| joint-7-30-5-30-3 | 9 | 32 | 71.9% | 9 | 32 | 71.9% | 3600.0 t | 9 | 32 | 71.9% | 3600.0 t | 9 | 25 | 64.0% | 3600.0 | 9 | 20 | 55.0% | 3600.0 | 9 | 14 | 35.7% | 3600.0 |
| joint-8-15-5-15-1 | 10 | 39 | 74.4% | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t | 10 | 15 | 33.3% | 3600.0 | 10 | 13 | 23.1% | 3600.0 | 10 | 13 | 23.1% | 3600.0 |
| joint-8-15-5-15-2 | 10 | 39 | 74.4% | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t | 10 | 18 | 44.4% | 3600.0 | 10 | 24 | 58.3% | 3600.0 | 10 | 17 | 41.2% | 3600.0 |
| joint-8-15-5-15-3 | 10 | 39 | 74.4% | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t | 10 | 14 | 28.6% | 3600.0 | 10 | 13 | 23.1% | 3600.0 | 10 | 13 | 23.1% | 3600.0 |
| joint-8-30-5-30-1 | 10 | 39 | 74.4% | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t | 10 | 32 | 68.8% | 3600.0 | 10 | 17 | 41.2% | 3600.0 | 10 | 17 | 41.2% | 3600.0 |
| joint-8-30-5-30-2 | 10 | 39 | 74.4% | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t | 10 | 23 | 56.5% | 3600.0 | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t |
| joint-8-30-5-30-3 | 10 | 39 | 74.4% | 10 | 39 | 74.4% | 3600.0 t | 10 | 39 | 74.4% | 3600.0 t | 10 | 22 | 54.5% | 3600.0 | 10 | 17 | 41.2% | 3600.0 | 10 | 17 | 41.2% | 3600.0 |
| joint-9-15-5-15-1 | 11 | 47 | 76.6% | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | 3600.0 t | 11 | 26 | 57.7% | 3600.0 | 11 | 15 | 26.7% | 3600.0 | 11 | 20 | 45.0% | 3600.0 |
| joint-9-15-5-15-2 | 11 | 47 | 76.6% | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | 3600.0 t | 11 | 27 | 59.3% | 3600.0 | 11 | 20 | 45.0% | 3600.0 | 11 | 20 | 45.0% | 3600.0 |
| joint-9-15-5-15-3 | 11 | 47 | 76.6% | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | 3600.0 t | 11 | 24 | 54.2% | 3600.0 | 11 | 15 | 26.7% | 3600.0 | 11 | 15 | 26.7% | 3600.0 |

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Table 5: Results for joint instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| joint-9-30-5-30-1 | 11 | 47 | 76.6% | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | 3600.0 t | 11 | 20 | 45.0% | 3600.0 | 11 | 20 | 45.0% | 3600.0 |
| joint-9-30-5-30-2 | 11 | 47 | 76.6% | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | — t | 11 | 46 | 76.1% | 3600.0 | 11 | 20 | 45.0% | 3600.0 | 11 | 20 | 45.0% | 3600.0 |
| joint-9-30-5-30-3 | 11 | 47 | 76.6% | 11 | 47 | 76.6% | 3600.0 t | 11 | 47 | 76.6% | 3600.0 t | 11 | 41 | 73.2% | 3600.0 | 11 | 15 | 26.7% | 3600.0 | 11 | 20 | 45.0% | 3600.0 |
| joint-10-15-5-15-1 | 11 | 56 | 80.4% | 11 | 56 | 80.4% | 3600.0 t | 11 | 56 | 80.4% | 3600.0 t | 11 | 44 | 75.0% | 3600.0 | 11 | 22 | 50.0% | 3600.0 | 11 | 33 | 66.7% | 3600.0 |
| joint-10-15-5-15-2 | 11 | 56 | 80.4% | 11 | 56 | 80.4% | 3600.0 t | 11 | 56 | 80.4% | 3600.0 t | 11 | 36 | 69.4% | 3600.0 | 11 | 22 | 50.0% | 3600.0 | 11 | 22 | 50.0% | 3600.0 |
| joint-10-15-5-15-3 | 11 | 56 | 80.4% | 11 | 56 | 80.4% | 3600.0 t | 11 | 56 | 80.4% | 3600.0 t | 11 | 21 | 47.6% | 3600.0 | 11 | 16 | 31.3% | 3600.0 | 11 | 22 | 50.0% | 3600.0 |
| joint-10-30-5-30-1 | 11 | 56 | 80.4% | 11 | 56 | 80.4% | 3600.0 t | 11 | 56 | 80.4% | — t | 11 | 56 | 80.4% | 3600.0 t | 11 | 22 | 50.0% | 3600.0 | 11 | 33 | 66.7% | 3600.0 |
| joint-10-30-5-30-2 | 11 | 56 | 80.4% | 11 | 56 | 80.4% | 3600.0 t | 11 | 56 | 80.4% | — t | 11 | 56 | 80.4% | 3600.0 t | 11 | 33 | 66.7% | 3600.0 | 11 | 33 | 66.7% | 3600.0 |
| joint-10-30-5-30-3 | 11 | 56 | 80.4% | 11 | 56 | 80.4% | 3600.0 t | 11 | 56 | 80.4% | — t | 11 | 56 | 80.4% | 3600.0 t | 11 | 33 | 66.7% | 3600.0 | 11 | 33 | 66.7% | 3600.0 |
| Average | 9.3 | 36.8 | 73.2% | 9.3 | 35.7 | 70.0% | 3600.0 | 9.3 | 35.4 | 68.4% | 3600.0 | 9.4 | 23.0 | 45.6% | 3402.9 | 9.4 | 16.1 | 34.1% | 3281.8 | 9.4 | 17.0 | 35.7% | 3249.1 |
| Solved | | | | | | 0 | | | | 0 | | | | 3 | | | | 4 | | | | 4 | |
| Trivial | | | | | | 32 | | | | 31 | | | | 4 | | | | 1 | | | | 1 | |
| Out of Memory | | | | | | 0 | | | | 4 | | | | 0 | | | | 0 | | | | 0 | |

Table 6: Results for random distance instances

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| drand-5-75-1 | 4 | 6 | 33.3% | 6 | 6 | 0.0% | 211.3 | 6 | 6 | 0.0% | 2.6 | 6 | 6 | 0.0% | 0.2 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.2 |
| drand-5-75-2 | 4 | 6 | 33.3% | 5 | 5 | 0.0% | 73.8 | 5 | 5 | 0.0% | 1.1 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.0 | 5 | 5 | 0.0% | 0.1 |
| drand-5-75-3 | 4 | 6 | 33.3% | 6 | 6 | 0.0% | 148.3 | 6 | 6 | 0.0% | 2.3 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.2 |
| drand-5-110-1 | 4 | 6 | 33.3% | 5 | 5 | 0.0% | 12.7 | 5 | 5 | 0.0% | 0.3 | 5 | 5 | 0.0% | 0.1 | 5 | 5 | 0.0% | 0.0 | 5 | 5 | 0.0% | 0.0 |
| drand-5-110-2 | 4 | 6 | 33.3% | 6 | 6 | 0.0% | 53.6 | 6 | 6 | 0.0% | 0.8 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.1 |
| drand-5-110-3 | 4 | 6 | 33.3% | 6 | 6 | 0.0% | 969.3 | 6 | 6 | 0.0% | 5.8 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.2 |
| drand-5-200-1 | 4 | 6 | 33.3% | 6 | 6 | 0.0% | 89.3 | 6 | 6 | 0.0% | 1.5 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.2 |
| drand-5-200-2 | 4 | 6 | 33.3% | 6 | 6 | 0.0% | 188.6 | 6 | 6 | 0.0% | 2.9 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.1 | 6 | 6 | 0.0% | 0.2 |
| drand-5-200-3 | 4 | 6 | 33.3% | 5 | 5 | 0.0% | 9.0 | 5 | 5 | 0.0% | 0.4 | 5 | 5 | 0.0% | 0.0 | 5 | 5 | 0.0% | 0.0 | 5 | 5 | 0.0% | 0.0 |

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Table 6: Results for random distance instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| drand-7-75-1 | 5 | 8 | 37.5% | 5 | 7 | 28.6% | 3600.0 | 5 | 7 | 28.6% | 3600.0 | 7 | 7 | 0.0% | 5.6 | 7 | 7 | 0.0% | 4.9 | 7 | 7 | 0.0% | 7.1 |
| drand-7-75-2 | 5 | 8 | 37.5% | 5 | 7 | 28.6% | 3600.0 | 5 | 6 | 16.7% | 3600.0 | 6 | 6 | 0.0% | 1.2 | 6 | 6 | 0.0% | 0.5 | 6 | 6 | 0.0% | 1.7 |
| drand-7-75-3 | 5 | 8 | 37.5% | 5 | 6 | 16.7% | 3600.0 | 5 | 6 | 16.7% | 3600.0 | 6 | 6 | 0.0% | 1.6 | 6 | 6 | 0.0% | 1.1 | 6 | 6 | 0.0% | 2.6 |
| drand-7-110-1 | 5 | 8 | 37.5% | 5 | 7 | 28.6% | 3600.0 | 5 | 7 | 28.6% | 3600.0 | 7 | 7 | 0.0% | 6.7 | 7 | 7 | 0.0% | 4.9 | 7 | 7 | 0.0% | 12.0 |
| drand-7-110-2 | 5 | 8 | 37.5% | 5 | 7 | 28.6% | 3600.0 | 5 | 7 | 28.6% | 3600.0 | 7 | 7 | 0.0% | 6.3 | 7 | 7 | 0.0% | 5.3 | 7 | 7 | 0.0% | 10.4 |
| drand-7-110-3 | 5 | 8 | 37.5% | 5 | 7 | 28.6% | 3600.0 | 5 | 7 | 28.6% | 3600.0 | 7 | 7 | 0.0% | 5.6 | 7 | 7 | 0.0% | 5.9 | 7 | 7 | 0.0% | 12.3 |
| drand-7-200-1 | 5 | 8 | 37.5% | 5 | 8 | 37.5% | 3600.0 t | 5 | 8 | 37.5% | 3600.0 t | 8 | 8 | 0.0% | 136.2 | 8 | 8 | 0.0% | 153.5 | 8 | 8 | 0.0% | 382.5 |
| drand-7-200-2 | 5 | 8 | 37.5% | 5 | 8 | 37.5% | 3600.0 t | 5 | 8 | 37.5% | 3600.0 t | 8 | 8 | 0.0% | 134.7 | 8 | 8 | 0.0% | 112.2 | 8 | 8 | 0.0% | 297.6 |
| drand-7-200-3 | 5 | 8 | 37.5% | 5 | 7 | 28.6% | 3600.0 | 5 | 7 | 28.6% | 3600.0 | 7 | 7 | 0.0% | 7.7 | 7 | 7 | 0.0% | 3.8 | 7 | 7 | 0.0% | 10.0 |
| drand-10-75-1 | 5 | 11 | 54.5% | 5 | 9 | 44.4% | 3600.0 | 5 | 7 | 28.6% | 3600.0 | 7 | 7 | 0.0% | 9.1 | 7 | 7 | 0.0% | 12.3 | 7 | 7 | 0.0% | 11.0 |
| drand-10-75-2 | 5 | 11 | 54.5% | 5 | 9 | 44.4% | 3600.0 | 5 | 9 | 44.4% | 3600.0 | 8 | 8 | 0.0% | 536.0 | 8 | 8 | 0.0% | 174.8 | 8 | 8 | 0.0% | 293.2 |
| drand-10-75-3 | 5 | 11 | 54.5% | 5 | 8 | 37.5% | 3600.0 | 5 | 8 | 37.5% | 3600.0 | 7 | 7 | 0.0% | 40.7 | 7 | 7 | 0.0% | 16.4 | 7 | 7 | 0.0% | 78.2 |
| drand-10-110-1 | 5 | 11 | 54.5% | 5 | 11 | 54.5% | 3600.0 t | 5 | 10 | 50.0% | 3600.0 | 8 | 8 | 0.0% | 567.3 | 8 | 8 | 0.0% | 369.1 | 8 | 8 | 0.0% | 1076.2 |
| drand-10-110-2 | 5 | 11 | 54.5% | 5 | 10 | 50.0% | 3600.0 | 5 | 11 | 54.5% | 3600.0 t | 8 | 8 | 0.0% | 1216.1 | 8 | 8 | 0.0% | 422.9 | 8 | 8 | 0.0% | 893.2 |
| drand-10-110-3 | 5 | 11 | 54.5% | 5 | 9 | 44.4% | 3600.0 | 5 | 9 | 44.4% | 3600.0 | 8 | 8 | 0.0% | 676.6 | 8 | 8 | 0.0% | 342.5 | 8 | 8 | 0.0% | 777.8 |
| drand-10-200-1 | 5 | 11 | 54.5% | 5 | 11 | 54.5% | 3600.0 t | 5 | 11 | 54.5% | 3600.0 t | 8 | 8 | 0.0% | 1919.9 | 8 | 8 | 0.0% | 456.7 | 8 | 8 | 0.0% | 1419.6 |
| drand-10-200-2 | 5 | 11 | 54.5% | 5 | 11 | 54.5% | 3600.0 t | 5 | 11 | 54.5% | 3600.0 t | 7 | 9 | 22.2% | 3600.0 | 5 | 11 | 54.5% | 3600.0 t | 5 | 11 | 54.5% | 3600.0 t |
| drand-10-200-3 | 5 | 11 | 54.5% | 5 | 9 | 44.4% | 3600.0 | 5 | 11 | 54.5% | 3600.0 t | 8 | 8 | 0.0% | 302.3 | 8 | 8 | 0.0% | 112.9 | 8 | 8 | 0.0% | 2494.0 |
| drand-15-75-1 | 6 | 16 | 62.5% | 6 | 15 | 60.0% | 3600.0 | 6 | 14 | 57.1% | 3600.0 | 7 | 10 | 30.0% | 3600.0 | 9 | 10 | 10.0% | 3600.0 | 6 | 11 | 45.5% | 3600.0 |
| drand-15-75-2 | 6 | 16 | 62.5% | 6 | 13 | 53.8% | 3600.0 | 6 | 14 | 57.1% | 3600.0 | 8 | 10 | 20.0% | 3600.0 | 9 | 9 | 0.0% | 1469.8 | 9 | 9 | 0.0% | 3295.2 |
| drand-15-75-3 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 7 | 9 | 22.2% | 3600.0 | 8 | 8 | 0.0% | 1577.4 | 8 | 8 | 0.0% | 2153.1 |
| drand-15-110-1 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 7 | 11 | 36.4% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 11 | 45.5% | 3600.0 |
| drand-15-110-2 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 7 | 11 | 36.4% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 11 | 45.5% | 3600.0 |
| drand-15-110-3 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 7 | 9 | 22.2% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 11 | 45.5% | 3600.0 |
| drand-15-200-1 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 6 | 13 | 53.8% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 16 | 62.5% | 3600.0 t |
| drand-15-200-2 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 6 | 14 | 57.1% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 11 | 45.5% | 3600.0 |
| drand-15-200-3 | 6 | 16 | 62.5% | 6 | 16 | 62.5% | 3600.0 t | 6 | 16 | 62.5% | 3600.0 t | 6 | 11 | 45.5% | 3600.0 | 6 | 11 | 45.5% | 3600.0 | 6 | 11 | 45.5% | 3600.0 |

*Continue on next page*

Table 6: Results for random distance instances – *Continued from previous page*

| | Trivial | | | QP | | | | RLT | | | | IP | | | | FEAS | | | | MAX | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Instance | LB | UB | Gap | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time | LB | UB | Gap | Time |
| drand-20-75-1 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 16 | 56.3% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-20-75-2 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 17 | 58.8% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 |
| drand-20-75-3 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 10 | 30.0% | 3600.0 |
| drand-20-110-1 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 15 | 53.3% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-20-110-2 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 13 | 46.2% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-20-110-3 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-20-200-1 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 18 | 61.1% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-20-200-2 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 18 | 61.1% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-20-200-3 | 7 | 21 | 66.7% | 7 | 21 | 66.7% | 3600.0 t | 7 | 21 | 66.7% | 3600.0 t | 7 | 17 | 58.8% | 3600.0 | 7 | 14 | 50.0% | 3600.0 | 7 | 14 | 50.0% | 3600.0 |
| drand-25-75-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 16 | 50.0% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-75-2 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 15 | 46.7% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 12 | 33.3% | 3600.0 |
| drand-25-75-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 16 | 50.0% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-110-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 20 | 60.0% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-110-2 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 20 | 60.0% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-110-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-200-1 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 21 | 61.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-200-2 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | — t | 8 | 24 | 66.7% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| drand-25-200-3 | 8 | 26 | 69.2% | 8 | 26 | 69.2% | 3600.0 t | 8 | 26 | 69.2% | 3600.0 t | 8 | 21 | 61.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 | 8 | 17 | 52.9% | 3600.0 |
| Average | 5.8 | 14.7 | 54.0% | 6.1 | 14.2 | 45.7% | 3032.5 | 6.1 | 14.1 | 45.4% | 2989.0 | 7.0 | 11.0 | 25.0% | 1969.9 | 7.0 | 10.4 | 23.4% | 1830.5 | 6.9 | 10.2 | 23.3% | 1978.3 |
| Solved | | | | | | 9 | | | | 9 | | | | 26 | | | | 28 | | | | 28 | |
| Trivial | | | | | | 30 | | | | 31 | | | | 0 | | | | 1 | | | | 2 | |
| Out of Memory | | | | | | 0 | | | | 1 | | | | 0 | | | | 0 | | | | 0 | |