Timing problems and vehicle routing

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GT Transport et Logistique, Toulouse, December the 5th, 2011
Context of this research

- General effort dedicated to better address “rich vehicle routing problems” involving many side constraints and attributes.

- Observation: several VRP settings deserve their richness to the temporal features they involve: TW, time-dependent, flexible, break scheduling...

- The same questions faced in different domains: vehicle routing, scheduling, PERT, and so on.

- That led us to build a large cross-domain analysis and classification of timing problems.
Several applications presenting similar *timing* issues

Timing features and problems
- Classification and notation
- Reductions

A timing feature example: soft time-windows

Timing re-optimization
Several problems

- Four problems originating from different domains

VRPTW

E/T scheduling

ship speed opt.

isotonic regression
Several problems

- Four problems originating from different domains:
  - VRPTW
  - E/T scheduling
  - ship speed opt.
  - isotonic regression

VRPTW

\[
\begin{align*}
\min_{(t_1,\ldots,t_n) \in \mathbb{R}^n^+} & \sum_{i=1}^{n} \left\{ \alpha (\bar{e}_i - t_i)^+ + \beta (t_i - \bar{l}_i)^+ \right\} \\
\text{s.t.} & \quad t_i + p_i + d_{i,i+1} \leq t_{i+1} \quad 1 \leq i < n \\
& \quad e_i \leq t_i \leq l_i \quad 1 \leq i \leq n
\end{align*}
\]
Several problems

- Four problems originating from different domains:
  - VRPTW
  - E/T scheduling
  - Ship speed opt.
  - Isotonic regression

Fixed Sequence

\[
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n^+} \sum_{i=1}^{n} \left( \epsilon_i(d_i - t_i)^+ + \tau_i(t_i - d_i)^+ \right)
\]

Optimizing tasks execution dates:

\[ s.t. \quad t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \]
Several problems

- Four problems originating from different domains:
  - VRPTW
  - E/T scheduling
  - ship speed opt.
  - isotonic regression

**Fixed Sequence**

\[
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n^+} \sum_{i=1}^{n} d_{i,i+1} \hat{c} \left( \frac{d_{i,i+1}}{t_{i+1} - t_i} \right)
\]

\[\begin{align*}
\text{s.t.} & \quad t_i + p_i + \frac{d_{i,i+1}}{v_{\text{max}}} \leq t_{i+1} \\
& \quad e_i \leq t_i \leq l_i \\
& \quad 1 \leq i \leq n
\end{align*}\]
Several problems

- Four problems originating from different domains:
  - VRPTW
  - E/T scheduling
  - ship speed opt.
  - isotonic regression

\[
\begin{align*}
\min_{t=(t_1,\ldots,t_n)} & \quad \|t - N\| \\
\text{s.t.} & \quad t_i \leq t_{i+1} \quad 1 \leq i < n
\end{align*}
\]
... with some characteristics in common

**VRPTW**

\[
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n^+} \sum_{i=1}^n \{\alpha (\bar{e}_i - t_i)^+ + \beta (t_i - \bar{l}_i)^+\}
\]

s.t. \( t_i + p_i + d_{i,i+1} \leq t_{i+1} \) \hspace{1cm} 1 \leq i < n
\( e_i \leq t_i \leq l_i \) \hspace{1cm} 1 \leq i \leq n

**E/T scheduling**

\[
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n^+} \sum_{i=1}^n \{e_i (d_i - t_i)^+ + \tau_i (t_i - d_i)^+\}
\]

s.t. \( t_i + p_i \leq t_{i+1} \) \hspace{1cm} 1 \leq i < n

**Isotonic regression**

\[
\min_{t=(t_1, \ldots, t_n)} \|t - N\|
\]

s.t. \( t_i \leq t_{i+1} \) \hspace{1cm} 1 \leq i < n

**Ship speed opt.**

\[
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n^+} \sum_{i=1}^n d_{i,i+1} \left( \frac{d_{i,i+1}}{t_{i+1} - t_i} \right)
\]

s.t. \( t_i + p_i + d_{i,i+1} / v_{max} \leq t_{i+1} \) \hspace{1cm} 1 \leq i < n
\( e_i \leq t_i \leq l_i \) \hspace{1cm} 1 \leq i \leq n

**TIMING**

\[
\min_{t=(t_1, \ldots, t_n) \in \mathbb{R}^n^+} \sum_{F_x \in \mathcal{F}_{OBJ}} \sum_{1 \leq y \leq m_x} \alpha_x \sum_{1 \leq y \leq m_x} f_y^x(t)
\]

s.t. \( t_i + p_i \leq t_{i+1} \) \hspace{1cm} 1 \leq i < n
\( f_y^x(t) \leq 0 \)
\( F_x \in \mathcal{F}_{CONS} \), \( 1 \leq y \leq m_x \)

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Timing problems seek to determine the execution dates \( (t_1, \ldots, t_n) \) for a sequence of activities.

- Totally ordered continuous variables
- Additional features \( F^x \) characterized by functions \( f_{y}^{x} \) for \( 1 \leq y \leq m_x \) that participate either in the objective or as constraints:
  - time windows, time-dependent proc. times, flexible travel times, time lags, no waiting, limited waiting, and so on...

\[
\begin{align*}
\min_{t=(t_1, \ldots, t_n) \in \mathbb{R}^n^+} & \quad \sum_{x \in \mathcal{F}_{\text{OBJ}}} \alpha_x \sum_{1 \leq y \leq m_x} f_x^{y}(t) \\
\text{s.t. } & \quad t_i + p_i \leq t_{i+1} \quad 1 \leq i < n \\
& \quad f_x^{y}(t) \leq 0 \quad F_x \in \mathcal{F}^{\text{CONS}}, \ 1 \leq y \leq m_x
\end{align*}
\]
Timing problems

Several names in the literature: *Scheduling, Timing, Projections onto Order Simplexes, Optimal service time problem* ...

Few dedicated studies, literature scattered among several research domains despite its relevance to many applications

Thus motivating a dedicated review and analysis of timing algorithms to fill the gap.
Timing features from the vehicle routing domain

- **Rich vehicle routing problems** can involve various *timing features*

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameters</th>
<th>Char. functions</th>
<th>Most frequent roles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>due dates $d_i$</td>
<td>$f_i(t) = (t_i - d_i)^+$</td>
<td>Service deadlines constraints, tardiness</td>
</tr>
<tr>
<td>$R$</td>
<td>release dates $r_i$</td>
<td>$f_i(t) = (r_i - t_i)^+$</td>
<td>Release-dates, earliness.</td>
</tr>
<tr>
<td>$TW$</td>
<td>time windows $TW_i = [e_i, l_i]$</td>
<td>$f_i(t) = (t_i - l_i)^+$</td>
<td>Time-window constraints, soft time windows.</td>
</tr>
<tr>
<td>$MTW$</td>
<td>multiple TW $MTW_i = \bigcup[e_{ik}, l_{ik}]$</td>
<td>$f_i(t) = \min_k [(t_i - l_{ik})^+ + (e_{ik} - t_i)^+]$</td>
<td>Multiple time-window constraints</td>
</tr>
<tr>
<td>$\Sigma c_i(t_i)$</td>
<td>general $c_i(t)$</td>
<td>$f_i(t) = c_i(t_i)$</td>
<td>Time-dependent service costs</td>
</tr>
<tr>
<td>$\Sigma c_i^{\text{CVX}}(t_i)$</td>
<td>convex $c_i^{\text{CVX}}(t_i)$</td>
<td>$f_i(t) = c_i^{\text{CVX}}(t_i)$</td>
<td>Time-d. convex service costs</td>
</tr>
<tr>
<td>$DUR$</td>
<td>total dur. $\delta_{\text{max}}$</td>
<td>$f(t) = (t_n - \delta_{\text{max}} - t_1)^+$</td>
<td>Duration or overall idle time</td>
</tr>
<tr>
<td>$NWT$</td>
<td>no wait</td>
<td>$f_i(t) = (t_{i+1} - p_i - t_i)^+$</td>
<td>No wait constraints</td>
</tr>
<tr>
<td>$IDL$</td>
<td>idle time $t_i$</td>
<td>$f_i(t) = (t_{i+1} - p_i - t_i - t_i)^+$</td>
<td>Limited idle time per stop, min idle time excess</td>
</tr>
<tr>
<td>$P(t)$</td>
<td>time-dependent proc. times $p_i(t_i)$</td>
<td>$f_i(t) = (t_i + p_i(t_i) - t_{i+1})^+$</td>
<td>Time-dependent driving-times</td>
</tr>
<tr>
<td>$TL$</td>
<td>time-lags $\delta_{ij}$</td>
<td>$f_i(t) = (t_j - \delta_{ij} - t_i)^+$</td>
<td>Time-lag constraints</td>
</tr>
<tr>
<td>$\Sigma c_i(\Delta t_i)$</td>
<td>general $c_i(t)$</td>
<td>$f_i(t) = c_i(t_{i+1} - t_i)$</td>
<td>Flexible travel times</td>
</tr>
<tr>
<td>$\Sigma c_{ij}(t_i, t_j)$</td>
<td>general $c_{ij}(t, t')$</td>
<td>$f_{ij}(t) = c_i(t_i, t_j)$</td>
<td>Separable objectives or constraints by any pairs of variables ...</td>
</tr>
</tbody>
</table>
These features can be classified and hierarchized (many-one linear reduction relationships between the associated timing problems).

Features in the NP-hard area lead to NP-hard timing problems.
In this presentation, brief glimpse of the analysis.

We examine a particular feature as illustrative example.

A similar study has been conducted on other features from this figure.
A feature example: soft time-windows

- Timing problem
  - with soft time-windows (penalized early and late arrival)
  - and generally with any convex separable cost

- We inventoried more than 30 algorithms from various domains (routing, scheduling, PERT, statistics...) that address these models.

- The solution block representation / active set framework (Chakravarti 1989, Best & Chakravarti 1990, Best et al. 2000, Ahuja & Orlin 2001) can be used to characterize these methods. But we need to generalize the optimality conditions to the non-smooth case.

\[
\begin{align*}
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n} & \sum_{i=1}^{n} \{ \alpha (\tilde{e}_i - t_i)^+ + \beta (t_i - \bar{l}_i)^+ \} \\
\text{s.t.} & \quad t_i + p_i \leq t_{i+1} \quad 1 \leq i < n
\end{align*}
\]

\[
\begin{align*}
\min_{(t_1, \ldots, t_n) \in \mathbb{R}^n} & \sum_{i=1}^{n} c_{CVX}^i(t_i) \\
\text{s.t.} & \quad t_i + p_i \leq t_{i+1}
\end{align*}
\]
A feature example: soft time-windows

- A block $B$ is defined as a subsequence of activities $(a_{B(1)}, ..., a_{B(|B|)})$ processed consecutively (such that $t_i + p_i = t_{i+1}$)

- Theorem (Generalization of Best & Chakravarti 1990):
  Let costs $c_i(t_i)$ be proper convex, eventually non-smooth, functions. A solution $(t^*_1, ..., t^*_n)$ of the timing problem with convex separable costs is optimal if and only if it can be assimilated to a succession of activity blocks $(B_1, ..., B_m)$ such that:

  1) **Blocks are optimally placed**: for each block $B_i$,
     \[ t^*_{B_i(1)} \in \text{argmin } C_{B_i}(t) \]

  2) **Blocks are spaced**: for each pair of blocks $(B_i, B_{i+1})$,
     \[ t^*_{B_i(1)} + \sum_j p_{B_i(j)} < t^*_{B_{i+1}(1)} \]

  3) **Blocks are consistent**: for each block $B_i$ and prefix block $B_i^k$,
     \[ \max \text{argmin } C_{B_i^k}(t) \geq t^*_{B_i(1)} \]
A feature example: soft time-windows

- Three main families of algorithms can be identified:
  - Primal feasible, that respect *spacing condition 2*
  - Dual feasible, that respect *consistency condition 3*
  - Dynamic programming

- To illustrate, consider this small problem with 6 activities

![Graph showing soft time-windows with 6 activities and their corresponding times and durations.](chart)
Primal feasible method, respecting the spacing condition.

- Brunk (1955): Minimum Lower Set Algorithm in $O(n^2)$ unimodal minimizations.
- Extended by Garey et al. (1988) and Best & Chakravarti (1990) to work, respectively, in $O(n \log n)$ elementary operations in the case of (E/T) scheduling, and $O(n)$ unimodal function minimizations in the general convex case.

Several other related methods designed for (E/T) scheduling

In the context of PERT with convex costs: Chrétienne and Sourd (2003)
A feature example: soft time-windows

- **Dual feasible method, respecting the consistency condition.**
  - Ayer et al. (1955): Pool Adjacent Violator Algorithm (PAV).
  - Extended to the general convex case by Best et al. 2000 and Ahuja & Orlin (2001) \(\rightarrow O(n)\) function minimizations
  - Can work in \(O(n \log^2 n)\) for Isotone Regression with \(||||_1\) (equivalent to \((E/T)\) with equal penalties for earliness and tardiness) (Pardalos 1995)
  - For the VRP with convex service costs, Dumas et al. 1990 can be viewed as another application of this principle
A feature example: soft time-windows


- Forward dynamic programming

\[
F_i(t) = \min_{0 \leq x \leq t} \{ c_i(x) + F_{i-1}(x - p_{i-1}) \}
\]

- Backward dynamic programming

\[
B_i(t) = \min_{x \geq t} \{ c_i(x) + B_{i+1}(x + p_i) \}
\]
Hence, many different methods for this particular feature example. The literature on timing problems is rich, but scattered. All in all, 26 different methods from different domains were classified as variations of 3 main algorithmic ideas.
Furthermore, when used within LS, solving all timing problems *from scratch* is generally not efficient.

General goal when exploring neighborhoods: solving series of timing problems on several activity permutations $\sigma \in \mathbb{N}$. 

$$\min_{t=(t_1,\ldots,t_n)\in\mathbb{R}^n+} \sum_{F^x \in \mathcal{F}_{OBJ}} \alpha_x \sum_{1 \leq y \leq m_x} f^x_y(t)$$

subject to

$$t_{\sigma^k(i)} + p_{\sigma^k(i)}\sigma^k(i+1) \leq t_{\sigma^k(i+1)}$$

$$f^x_y(t) \leq 0$$
Timing re-optimization

- In classical VRP neighborhoods, the neighborhood size is often rather large: $|N| = \Omega(n^2)$, and permutations are very particular.
  - They have a bounded number (often $\leq 4$) of breakpoints: integers $x$ such that $\sigma(x)+1 \neq \sigma(x+1)$,

- The resulting sequences of activities can be assimilated to a recombination of a small number of subsequences.

![Diagram of sequences A, B, C, D with arrows indicating recombination process.](image-url)
Efficient *timing re-optimization* by means of a subset of 4 procedures, used within local searches:

- Initialization of suitable re-optimization data for a single activity
- Forward (F) or backward (B) computation of data on larger subsequences
- Evaluation of a concatenation of two (C2) or more (C3+) subsequences

**Algorithm 1** Re-optimization

1. Build re-optimization data on subsequences of the *incumbent timing problem* $\mathcal{T}$, using *initialize*, and *forward extension* or *backward extension*.
2. For each timing subproblem $\mathcal{T}^k$, $k \in \{1, \ldots, N\}$;
3. Determine the breakpoints involved in the permutation function $\sigma^k$;
4. Evaluate the optimal cost of $\mathcal{T}^k$, as the concatenation of $b(\sigma) + 1$ activity subsequences from $\mathcal{T}$ (see Equation 39).
Example of soft time-windows: Forward and backward extension to compute data on subsequences, and evaluate concatenation of 2 sequences (Ibaraki et al. 2005, 2008):

$$Z^*(A_1 \oplus A_2) = \min_{t \geq 0} \left\{ F(A_1)(t) + B(A_2)(t + p_{A_1(|A_1|)A_2(1)}) \right\}$$

- In the convex case, the concatenation of 3+ sequences is also addressed efficiently.
- $O(\log \phi)$ for convex piecewise functions with a total of $\phi$ pieces.
- $O(\log n)$ move evaluations for soft TW
Conclusions of this analysis

For other features: Surveying the literature, we classified many re-optimization based methodologies from various domains, and for a large variety of attributes. (Savelsbergh 1985, 1992, Kindervater and Savelsbergh 1997, Campbell and Savelsbergh 2004, Ergun and Orlin 2006, Irnich 2008, Hashimoto et al. 2006, 2008, Kedad-Sidhoum and Sourd 2010)...

We could identify a set of state-of-the-art timing methods, which are the key to solve many rich VRP settings:
Conclusions of this analysis

<table>
<thead>
<tr>
<th>Problem</th>
<th>From Scratch</th>
<th>Re-opt. by concat.</th>
<th>F/B</th>
<th>C2</th>
<th>C3+</th>
<th>Sd</th>
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<td>( { W</td>
<td>\sigma } )</td>
<td>Min idle time</td>
<td>( O(n) )</td>
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<td>( \sigma</td>
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<td>{ D</td>
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<td>\sigma }</td>
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<td>Cordeau and Laporte (2003)</td>
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<td>{ \Sigma c_i^{\text{cvs}}(t_j - t_i), \Sigma c_i^{\text{cvs}}(t_i)</td>
<td>\sigma }</td>
<td>Ahuja et al. (2003)</td>
<td>( O(n^3 \log n \log(nU)) )</td>
<td></td>
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</tr>
</tbody>
</table>

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Conclusions of this analysis

- Large analysis of a rich body of problems with time characteristics and totally ordered variables. Cross-domain synthesis, considering methods from various fields such as vehicle routing, scheduling, PERT, and isotonic regression. Identification of main resolution principles.

- For several “rich” combinatorial optimization settings, the timing sub-problems represent the core of “richness” and deserve particular attention.

- Furthermore, timing sub-problems frequently arise in the context of local search, and thus we analyzed both stand-alone resolution and efficient solving of series of problems.
Perspectives

- Timing procedures are being integrated in a recent efficient Hybrid Genetic Search with Advanced Diversity Control (HGSADVC- Vidal et al. 2011), opening the way to a new generalist solver for rich VRPs with timing features.

- Several features and feature combinations were identified in this work, for which new timing algorithms (including re-optimization procedures) should be sought.

- Generalization to other cumulative resources, multi-objective or stochastic settings.

- More studies on complexity lower bounds.

Thanks a lot for your attention
Bibliography

Bibliography

Bibliography